On the Capacity of MIMO Relay Channels

Bo Wang, *Student Member, IEEE*, Junshan Zhang, *Member, IEEE*, and Anders Høst-Madsen, *Senior Member, IEEE*

***Abstract*—We study the capacity of multiple-input multiple-output (MIMO) relay channels. We first consider the Gaussian MIMO relay channel with fixed channel conditions, and derive upper bounds and lower bounds that can be obtained numerically by convex programming. We present algorithms to compute the bounds. Next, we generalize the study to the Rayleigh fading case. We find an upper bound and a lower bound on the ergodic capacity. It is somewhat surprising that the upper bound can meet the lower bound under certain regularity conditions (not necessarily degradedness), and therefore the capacity can be characterized exactly; previously this has been proven only for the degraded Gaussian relay channel. We investigate sufficient conditions for achieving the ergodic capacity; and in particular, for the case where all nodes have the same number of antennas, the capacity can be achieved under certain signal-to-noise ratio (SNR) conditions. Numerical results are also provided to illustrate the bounds on the ergodic capacity of the MIMO relay channel over Rayleigh fading. Finally, we present a potential application of the MIMO relay channel for cooperative communications in *ad hoc* networks.**

***Index Terms*—Cooperative communications, ergodic capacity, multiple-input multiple-output (MIMO), relay channel.**

I. Introduction

Wireless networking constitutes an important component of future information technology applications. Recently, the use of multiple antennas at wireless transmitters and receivers has been identified as an enabling technique for high-rate multimedia transmissions over wireless channels. Although the point-to-point multiple-input multiple-output (MIMO) channel is relatively well understood (particularly the information-theoretic aspects therein), the general area of multiuser MIMO communications, i.e., the field of communications involving a network of many users using multiple antennas, is still at a stage of its infancy and poses a rich set of challenges to the research community.

We consider MIMO relay channels because this application has great potential in wireless networks. For example, for transmissions from a base station (access point) to users, relay stations can be exploited to relay messages for end users. The motivation for using relay stations can be simply put as follows. 1) In a cellular network, direct transmissions between the base station and users close to the cell boundary can be very expensive in terms of the transmission power required to ensure reliable communications; and 2) existing RF technologies typically can accommodate only one or two antennas at the user end, indicating that current wireless systems cannot fully benefit from promising space–time techniques. By making use of relay stations (which can accommodate multiple antennas) to relay the message, the channel is effectively converted into a MIMO relay channel. Another example is to utilize relay nodes for cooperative communications in *ad hoc* networks, where the nodes close to the active transmitter and the receiver can relay data packets from the transmitter to the receiver. In a nutshell, relaying in wireless networks is garnering much attention (see, e.g., [18], [31], [12], [27], [10], [17]). Notably, some interesting cooperative schemes have recently been investigated (see [14], [15], [21], [22], and the references therein).

In this paper, we study the channel capacity of MIMO relay channels. As illustrated in Fig. 1, a three-terminal relay channel can be viewed as a hybrid of a broadcast channel (BC) and a multiple-access channel (MAC). For the relay node, a key assumption is that it works in a full-duplex mode. In a MIMO relay channel, both BC and MAC parts are vector channels, making it nontrivial to derive bounds on the channel capacity. Indeed, because the vector BC is not degraded in general, the corresponding channel capacity region is challenging to characterize. Notably, this problem has been solved by very recent work [29]. It has also been studied in [25] by using the duality technique, which establishes a dual relationship between the Gaussian MAC vector channel and the BC vector channel. The sum rate of the Gaussian vector BC channel has been studied in [26], [32], [2]. The capacity region of the MAC channel is relatively better understood (see, e.g., [30], [3], [20], [33]).

We first consider capacity bounds on the Gaussian MIMO relay channel with fixed channel gains. We derive upper bounds and lower bounds, and discuss the corresponding codebook structures. We note that the upper bound for a general relay channel in [4] involves maximization over the joint (multidimensional) distribution of the codebooks at the source node and the relay node, and in general, the corresponding characterization for vector channels is highly nontrivial. In this paper, we derive an upper bound involving maximization over two covariance matrices and one scalar parameter ρ. Loosely speaking, parameter ρ "captures" the cooperation between the transmitted signals from the source node and the relay node. Moreover, using ρ leads to a simplified upper bound and enables us to solve the maximization problem by convex programming. Next, we give a lower bound by finding the maximum between the capacity for the direct link channel (from the source node to the destination node) and that for the cascaded channel (from the source node to the relay node and from the relay node to

Manuscript received March 2, 2004; revised July 20, 2004. This work was supported in part by the National Science Foundation under Grant ANI-0208135 and under a grant from the Intel Research Council. The material in this paper was presented in part at the 41st Allerton Conference on Communications, Control, and Computing, Monticello, IL, October 2003 and at the 38th Conference on Information Sciences and Systems, Princeton, NJ, March 2004.

B. Wang and J. Zhang are with the Department of Electrical Engineering, Arizona State University, Tempe, AZ 85287 USA (e-mail: bo.wang@asu.edu; junshan.zhang@asu.edu).

A. Høst-Madsen is with the Department of Electrical Engineering, University of Hawaii, Honolulu, HI 96822 USA (e-mail: madsen@spectra.eng.hawaii.edu).

Communicated by R. Müller, Associate Editor for Communications.

Digital Object Identifier 10.1109/TIT.2004.839487

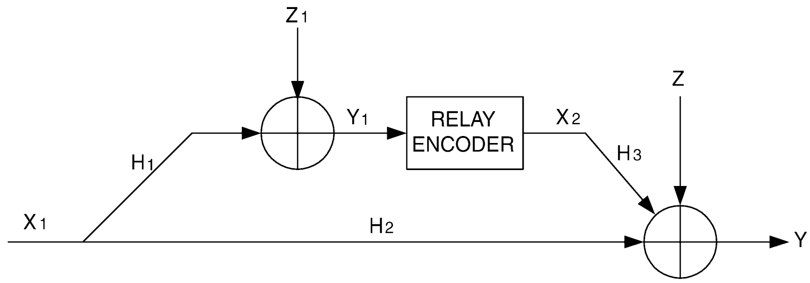

Fig. 1. MIMO relay channel.

the destination node). We present algorithms to compute the bounds accordingly. We also provide in Appendix B another lower bound by using the water-filling technique.

Next, we generalize the study to a more interesting case—the Rayleigh fading case. We focus on the ergodic capacity of the MIMO relay channel, assuming receiver channel state information (CSI) only. It is somewhat surprising that the upper bound can meet the lower bound under certain conditions (not necessarily degradedness), indicating that the ergodic capacity can be characterized exactly. Thus motivated, we investigate conditions for capacity achievability. In particular, we identify sufficient conditions to achieve the ergodic capacity when all nodes have the same number of antennas; and our intuition for this finding is as follows. The source node and the relay node can function as a "virtual" transmit antenna array when the relay node is located close to the source node, thus making it possible to achieve the capacity.

We note that the findings on the ergodic capacity point to independent coding strategies at the source node and the relay node. Such independence of coding strategies is due to the channel uncertainty (randomness) at the transmitters. Compared to the direct link, the relay channel offers a significant capacity gain, thanks to the multiple-access gain from the MAC part and the multiuser broadcast gain from the BC part. We note that both the multiple-access gain and the broadcast gain benefit from the full duplexity at the relay node. Needless to say, a key step to reap the capacity gain is to develop coding strategies for the cooperative MAC and the cooperative BC. We also provide numerical examples to illustrate the upper bound and the lower bound on the ergodic capacity. Motivated by the capacity gain by using the relay node, we finally discuss the utility of the MIMO relay channel in cooperative communications in *ad hoc* networks.

During the final stage of our preparation for this paper, we were informed of independent work [13], which presents an elegant proof for the independence of the signals from the source node and the relay node for the fading case. Building on this result and our preliminary works [27], [28], we have obtained the results in Theorems 4.1 and 4.2. As noted above, we have also investigated in depth the sufficient conditions for capacity achievability for two interesting cases, i.e., the high-signal–to–noise-ratio (SNR) regime and the scalar case. Loosely speaking, our capacity results for the Rayleigh fading case can be viewed as a generalization of Theorem 2 in [12].

Throughout this paper, we use $\mathbb{E}$ to denote the expectation operator (in some cases, subscripts are used to specify the random variable); "$\dagger$" stands for the conjugate transpose; $\boldsymbol{I}_M$ denotes an identity matrix; $\mathbf{0}$ is an all-zero matrix of proper dimensions; the distribution of a circularly symmetric complex Gaussian vector with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ is denoted as $\mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$; $\preceq$, $\prec$, $\succ$, and $\succeq$ are used in the matrix positive (semi)definite ordering sense [11].

The rest of the paper is organized as follows. In Section II, we present the system model. We derive in Section III upper bounds and lower bounds on the capacity of the Gaussian MIMO relay channel with fixed channel gains. Next, we generalize the study to the Rayleigh fading case. We present in Section IV an upper bound and a lower bound on the ergodic capacity and give numerical results for different SNR cases. We then discuss sufficient conditions for achieving the ergodic capacity in Section V. Finally, a potential application of the relay channel in cooperative communications in ad hoc networks is discussed in Section VI.

II. System Model

Consider a general MIMO relay channel where the received signals at the relay and destination nodes can be written as

$$\begin{cases} \boldsymbol{Y}_1 &= \sqrt{\eta_1}\boldsymbol{H}_1\boldsymbol{X}_1 + \boldsymbol{Z}_1 \\ \boldsymbol{Y} &= \sqrt{\eta_2}\boldsymbol{H}_2\boldsymbol{X}_1 + \sqrt{\eta_3}\boldsymbol{H}_3\boldsymbol{X}_2 + \boldsymbol{Z} \end{cases} \tag{1}$$

where

- $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ are $M_1 \times 1$ and $M_2 \times 1$ transmitted signals from the source node and the relay node; the power constraints on the transmit signals are $\mathbb{E}(\boldsymbol{X}_1^\dagger \boldsymbol{X}_1) \leq M_1$ and $\mathbb{E}(\boldsymbol{X}_2^\dagger \boldsymbol{X}_2) \leq M_2$;
- $\boldsymbol{Y}$ and $\boldsymbol{Y}_1$ are $N \times 1$ and $N_1 \times 1$ received signals at the destination node and the relay node. We assume that
  - the relay node has two sets of antennas, one for reception and the other for transmission. That is, the relay node operates in the full-duplex mode;
  - since the relay node has full knowledge of what to transmit therein, it can cancel out the interference from its own transmit antennas at its receive antennas.
- $\boldsymbol{H}_1$, $\boldsymbol{H}_2$, and $\boldsymbol{H}_3$ are $N_1 \times M_1$, $N \times M_1$, and $N \times M_2$ channel gain matrices, as depicted in Fig. 1. In what follows, we consider two scenarios for the channel matrices:
  - all the channel matrices are fixed and known at both the transmitters and the receivers;
  - all the channel matrices are random and independent, where the entries of each matrix are independent and

identically distributed (i.i.d.) complex Gaussian variables with zero mean, independent real and imaginary parts, each with variance $1/2$, and they are available at the corresponding receivers only (i.e., receiver CSI only).

- η_1, η_2, and η_3 are parameters related to the SNR [16]

$$\eta_1 = \frac{\mathrm{SNR}_1}{M_1}, \quad \eta_2 = \frac{\mathrm{SNR}_2}{M_1}, \quad \eta_3 = \frac{\mathrm{SNR}_3}{M_2} \tag{2}$$

  where SNR_1 and SNR_2 are the normalized power ratios of $\boldsymbol{X}_1$ to the noise (after fading) at each receiver antenna of the relay node and the destination node, and SNR_3 is the normalized power ratio of $\boldsymbol{X}_2$ to the noise at each antenna of the destination node;

- $\boldsymbol{Z}$ and $\boldsymbol{Z}_1$ are independent $N \times 1$ and $N_1 \times 1$ circularly symmetric complex Gaussian noise vectors with distributions $\mathcal{CN}(\mathbf{0},\ \boldsymbol{I}_N)$ and $\mathcal{CN}(\mathbf{0},\ \boldsymbol{I}_{N_1})$, and are uncorrelated to $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$.

III. Capacity Bounds: The Fixed Channel Case

In the following, we derive upper bounds and lower bounds on the capacity of the Gaussian MIMO relay channel with fixed channel gains.

A. Upper Bounds and Lower Bounds

Recall from [4] that the channel capacity of a general Gaussian relay channel is upper-bounded by

$$C^G \le \max_{p(\boldsymbol{x}_1, \boldsymbol{x}_2)} \min\left(\boldsymbol{I}(\boldsymbol{X}_1; \boldsymbol{Y}, \boldsymbol{Y}_1 | \boldsymbol{X}_2), \boldsymbol{I}(\boldsymbol{X}_1, \boldsymbol{X}_2; \boldsymbol{Y})\right) \tag{3}$$

where the first term in $\min(\cdot, \cdot)$ can be treated as the sum rate from the source node to the relay node and the destination node, corresponding to a BC part; and the second term can be viewed as the sum rate from the source node and the relay node to the destination node, corresponding to a MAC part. Indeed, (3) has an interesting max-flow min-cut interpretation [5], as illustrated in Fig. 2. Roughly speaking, the rate of the information flow transmitted on the relay channel is constrained by the bottleneck corresponding to either the first cut (BC) or the second one (MAC).

Without loss of generality, let $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ be random vectors with zero-mean and covariance matrices $\boldsymbol{\Sigma}_{ij}$, defined as $\boldsymbol{\Sigma}_{ij} = \mathbb{E}[\boldsymbol{X}_i \boldsymbol{X}_j^\dagger]$ for $i, j = 1, 2$. Throughout, we assume that $\det(\boldsymbol{\Sigma}_{22}) > 0$ and $\det(\boldsymbol{\Sigma}_{11}) > 0$. Define $\boldsymbol{A} \triangleq \boldsymbol{\Sigma}_{11}^{-\frac{1}{2}} \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-\frac{1}{2}}$. First, we need the following lemmas (the proof of Lemma 3.1 is relegated to Appendix A).

Lemma 3.1: There exists $\rho \in [0, 1]$ such that

$$\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^\dagger \preceq (1 - \rho^2) \boldsymbol{I}_{M_1} \tag{4}$$

and the equality can be achieved by a matrix $\boldsymbol{\Sigma}_{12} \in \mathcal{C}^{M_1 \times M_2}$ when $M_1 \le M_2$.

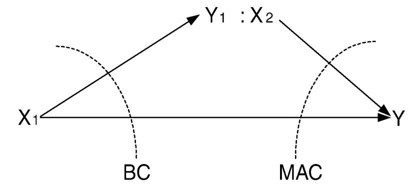

Fig. 2. The relay channel: max-flow min-cut.

Intuitively speaking, Lemma 3.1 reveals that for any given $\boldsymbol{\Sigma}_{11}$ and $\boldsymbol{\Sigma}_{22}$, we can find codebooks for $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ such that the covariance matrix $\boldsymbol{\Sigma}_{12}$ satisfies the following inequality:

$$\mathbf{0} \preceq \boldsymbol{\Sigma}_{11}^{-\frac{1}{2}} \left(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\Sigma}_{21} \right) \boldsymbol{\Sigma}_{11}^{-\frac{1}{2}} \preceq (1 - \rho^2) \boldsymbol{I}_{M_1}.$$

Furthermore, if $M_1 \le M_2$, then for any given $\rho \in [0, 1]$, there exists $\boldsymbol{\Sigma}_{12}$ such that the equality is achieved.

Lemma 3.2: For any two complex random vectors $\boldsymbol{V}$ and $\boldsymbol{W}$, $\forall\, a > 0$, we have that

$$\mathbb{E}(\boldsymbol{V}\boldsymbol{W}^\dagger + \boldsymbol{W}\boldsymbol{V}^\dagger) \preceq \mathbb{E}\left(\frac{1}{a} \boldsymbol{V}\boldsymbol{V}^\dagger + a \boldsymbol{W}\boldsymbol{W}^\dagger \right) \tag{5}$$

the equality is achieved if $\boldsymbol{V} = a\boldsymbol{W}$.

The proof follows simply from the fact that the covariance matrix of a random vector $(\boldsymbol{V} - a\boldsymbol{W})$ is always positive semidefinite. That is,

$$\begin{aligned} \mathrm{Cov}(\boldsymbol{V} - a\boldsymbol{W}) &= \mathbb{E}\left(\boldsymbol{V}\boldsymbol{V}^\dagger + a^2 \boldsymbol{W}\boldsymbol{W}^\dagger \right) \\ &\quad - a\mathbb{E}\left(\boldsymbol{V}\boldsymbol{W}^\dagger + \boldsymbol{W}\boldsymbol{V}^\dagger \right) \\ &\succeq \mathbf{0}. \end{aligned}$$

Let $\boldsymbol{V} = \sqrt{\eta_2} \boldsymbol{H}_2 \boldsymbol{\Sigma}_{11}^{\frac{1}{2}} \boldsymbol{A} \boldsymbol{\Sigma}_{22}^{-\frac{1}{2}} \boldsymbol{X}_2$ and $\boldsymbol{W} = \sqrt{\eta_3} \boldsymbol{H}_3 \boldsymbol{X}_2$. It follows that

$$\begin{aligned} \mathbb{E}(\boldsymbol{V}\boldsymbol{W}^\dagger + \boldsymbol{W}\boldsymbol{V}^\dagger) &= \sqrt{\eta_2 \eta_3} \boldsymbol{H}_2 \boldsymbol{\Sigma}_{12} \boldsymbol{H}_3^\dagger \\ &\quad + \sqrt{\eta_2 \eta_3} \left(\boldsymbol{H}_2 \boldsymbol{\Sigma}_{12} \boldsymbol{H}_3^\dagger \right)^\dagger \\ \mathbb{E}(\boldsymbol{V}\boldsymbol{V}^\dagger + \boldsymbol{W}\boldsymbol{W}^\dagger) &= \eta_2 \boldsymbol{H}_2 \boldsymbol{\Sigma}_{11}^{\frac{1}{2}} \boldsymbol{A}\boldsymbol{A}^\dagger \boldsymbol{\Sigma}_{11}^{\frac{1}{2}} \boldsymbol{H}_2^\dagger \\ &\quad + \eta_3 \boldsymbol{H}_3 \boldsymbol{\Sigma}_{22} \boldsymbol{H}_3^\dagger. \end{aligned}$$

If signals $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are chosen such that $\boldsymbol{A}\boldsymbol{A}^\dagger = \rho^2 \boldsymbol{I}_{M_1}$, then

$$\mathbb{E}(\boldsymbol{V}\boldsymbol{V}^\dagger + \boldsymbol{W}\boldsymbol{W}^\dagger) = \rho^2 \eta_2 \boldsymbol{H}_2 \boldsymbol{\Sigma}_{11} \boldsymbol{H}_2^\dagger + \eta_3 \boldsymbol{H}_3 \boldsymbol{\Sigma}_{22} \boldsymbol{H}_3^\dagger.$$

Applying Lemma 3.2, for $\forall\, a > 0$, we have that

$$\begin{aligned} &\sqrt{\eta_2 \eta_3} \boldsymbol{H}_2 \boldsymbol{\Sigma}_{12} \boldsymbol{H}_3^\dagger + \sqrt{\eta_2 \eta_3} \left(\boldsymbol{H}_2 \boldsymbol{\Sigma}_{12} \boldsymbol{H}_3^\dagger \right)^\dagger \\ &\qquad \preceq \frac{\rho^2}{a} \eta_2 \boldsymbol{H}_2 \boldsymbol{\Sigma}_{11} \boldsymbol{H}_2^\dagger + a \eta_3 \boldsymbol{H}_3 \boldsymbol{\Sigma}_{22} \boldsymbol{H}_3^\dagger. \end{aligned} \tag{6}$$

It is clear that the optimal distribution $p(\boldsymbol{x}_1, \boldsymbol{x}_2)$ in (3) is Gaussian [4], [24] (see, e.g., Step (b) following (19) and Step (a) following (22)). Observe that if Gaussian codebooks are applied, the maximization problem on the RHS of (3) would be with respected to three covariance matrices $\boldsymbol{\Sigma}_{11}$,

$\mathbf{\Sigma}_{22}$, and $\mathbf{\Sigma}_{12}$; and this is nonconvex and highly nontrivial in general. In Lemmas 3.1 and 3.2, we use one parameter ρ to capture the correlation between $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ (in contrast to the cross-covariance matrix $\mathbf{\Sigma}_{12}$), and this enables us to solve the optimization by convex programming techniques (in Appendix A, we present the definition for ρ). In what follows, we present our results for the upper bound on the capacity of Gaussian MIMO relay channels.

Theorem 3.1: **[Fixed Channel Case]** An upper bound on the capacity of the MIMO relay channel is given by

$$C^G \leq C^G_{\text{upper}} = \max_{0\leq\rho\leq 1,\, \mathbf{\Sigma}_{11}, \mathbf{\Sigma}_{22}} \min\left(C^G_1,\, C^G_2\right) \tag{7}$$

where $\text{tr}(\mathbf{\Sigma}_{11}) \leq M_1$, $\text{tr}(\mathbf{\Sigma}_{22}) \leq M_2$; C^G_1 and C^G_2 are given by

$$C^G_1 \triangleq \log\left[\det\left(\boldsymbol{I}_{M_1} + \left(1-\rho^2\right)\begin{bmatrix}\sqrt{\eta_1}\boldsymbol{H}_1\\ \sqrt{\eta_2}\boldsymbol{H}_2\end{bmatrix} \times \mathbf{\Sigma}_{11}\begin{bmatrix}\sqrt{\eta_1}\boldsymbol{H}_1\\ \sqrt{\eta_2}\boldsymbol{H}_2\end{bmatrix}^\dagger\right)\right] \tag{8}$$

$$C^G_2 \triangleq \inf_{a>0}\log\left[\det\left(\boldsymbol{I}_N + \left(\eta_2 + \frac{\rho^2}{a}\sqrt{\eta_2\eta_3}\right)\boldsymbol{H}_2\mathbf{\Sigma}_{11}\boldsymbol{H}_2^\dagger + \left(\eta_3 + a\sqrt{\eta_2\eta_3}\right)\boldsymbol{H}_3\mathbf{\Sigma}_{22}\boldsymbol{H}_3^\dagger\right)\right]. \tag{9}$$

Remarks: To find the upper bound, we need to characterize the optimal input covariance matrices $\mathbf{\Sigma}_{11}$ and $\mathbf{\Sigma}_{22}$. Define

$$f_0(\mathbf{\Sigma}_{11},\, \mathbf{\Sigma}_{22},\, \rho) \triangleq \min\left(C^G_1\,, C^G_2\right).$$

It can be seen that C^G_1 is concave in $\mathbf{\Sigma}_{11}$ and that C^G_2 is concave in $(\mathbf{\Sigma}_{11}, \mathbf{\Sigma}_{22})$. It follows that $\min(C^G_1, C^G_2)$ is concave in $(\mathbf{\Sigma}_{11}, \mathbf{\Sigma}_{22})$ [1]. More precisely, for a given ρ, the upper bound is concave in $(\mathbf{\Sigma}_{11}, \mathbf{\Sigma}_{22})$; and it can be found by convex programming. In what follows, we present an algorithm to compute the upper bound.

Algorithm I

1. Carry out a quantization of interval $[0,1]$, and denote the corresponding set of (ascending) values as $\{\rho_1, \rho_2, \ldots, \rho_m\}$;
2. For a given ρ_i, apply convex programming to find the optimal value of $f_0(\cdot,\cdot)$ and corresponding optimal $\mathbf{\Sigma}_{11}$ and $\mathbf{\Sigma}_{22}$, by solving the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & f_0(\mathbf{\Sigma}_{11},\, \mathbf{\Sigma}_{22}) \\ \text{subject to} \quad & \text{tr}(\mathbf{\Sigma}_{11}) - M_1 \leq 0, \quad \text{tr}(\mathbf{\Sigma}_{22}) - M_2 \leq 0 \\ & \mathbf{\Sigma}_{11} \succeq \mathbf{0}, \quad \mathbf{\Sigma}_{22} \succeq \mathbf{0}; \end{aligned}$$

3. Pick another ρ_i and repeat Step 2. Go to the next step when the set $\{\rho_1, \rho_2, \ldots, \rho_m\}$ is exhausted;

Algorithm I *(Continued)*

4. Compare all the values of $f_0(\cdot,\cdot)$ associated with $\{\rho_1, \rho_2, \ldots, \rho_m\}$, and find the largest and identify the corresponding optimal parameters ρ^*_k, $\mathbf{\Sigma}_{11}$ and $\mathbf{\Sigma}_{22}$;
5. Quantize $\left[\rho^*_{k-1}, \rho^*_{k+1}\right]$ and go to Step 2 using

$$\{\rho^*_{k-1} + \Delta\rho_1, \rho^*_{k-1} + \Delta\rho_2, \ldots, \rho^*_{k-1} + \Delta\rho_m\}$$

   for a new search;
6. Compare the refined results from the new search with the old ones. If the error requirement ϵ is met, end the procedures; otherwise, go to previous steps for another new search.

A few more words on Algorithm I. Since there is no *a priori* information about ρ in the initialization step, the quantization is equal-span. After the first iteration, we choose the "best guess" of ρ (namely, ρ^*_k), and then refine the search around it. To guarantee the convergency to the optimal point, the quantization level m should be reasonably large (e.g., $m \geq 10$).

In the above, we use ρ to capture the correlation between signals transmitted from the source node and the relay node. Now, we discuss the structure of the corresponding codebooks. We can rewrite the transmitted signal $\boldsymbol{X}_1$ as

$$\boldsymbol{X}_1 = \widetilde{\boldsymbol{X}}_{10} + \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\boldsymbol{X}_2$$

with $\widetilde{\boldsymbol{X}}_{10} \triangleq \boldsymbol{X}_1 - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\boldsymbol{X}_2$. Observe that $\widetilde{\boldsymbol{X}}_{10}$ is independent of $\boldsymbol{X}_2$. Intuitively speaking, the vector signal $\boldsymbol{X}_1$ can be decomposed into two orthogonal parts, which are independent of each other. The second part, $\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\boldsymbol{X}_2$, stands for the projection of $\boldsymbol{X}_1$ onto the direction of $\boldsymbol{X}_2$. Thus, $\boldsymbol{X}_1 = \widetilde{\boldsymbol{X}}_{10} + \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\boldsymbol{X}_2$ can be viewed as a generalization of [4, Theorem 5].

For the special case where all the channel coefficients are scalars (denoted as h_1, h_2, and h_3, respectively), we have that $\Sigma_{11} = \Sigma_{22} = 1$. Accordingly, we have that

$$C^G_1 = \log\left[1 + (1-\rho^2)\left(\eta_1|h_1|^2 + \eta_2|h_2|^2\right)\right]$$

and

$$\log\left[1 + \eta_2|h_2|^2 + \eta_3|h_3|^2 + \sqrt{\eta_2\eta_3}\left(\frac{\rho^2}{a}|h_2|^2 + a|h_3|^2\right)\right] \geq \log\left[1 + \eta_2|h_2|^2 + \eta_3|h_3|^2 + 2\rho\sqrt{\eta_2\eta_3}|h_2||h_3|\right]. \tag{10}$$

It follows that

$$C^G_2 = \log\left[1 + \eta_2|h_2|^2 + \eta_3|h_3|^2 + 2\rho\sqrt{\eta_2\eta_3}|h_2||h_3|\right] \tag{11}$$

which boils down to a result in [6].

Consider channel models where the relay node may or may not be used to aid the transmissions. If not used, the channel becomes a point-to-point Gaussian MIMO channel. On the other hand, if the relay node is used to aid the transmission and the destination node treats the information directly from the source node as noise, the channel boils down to a cascaded channel. We have the following lower bound by finding the maximum between the information rates for the two channel models.

Theorem 3.2: **[Fixed Channel Case]** A lower bound on the capacity of the Gaussian MIMO relay channel is given by

$$C^G \geq C^G_{\text{lower}} = \max\left(C^G_d,\ \min\left(C^G_3,\ C^G_4\right)\right) \quad (12)$$

where

$$C^G_d \triangleq \max_{\boldsymbol{\Sigma}_{11}} \log\left[\det\left(\boldsymbol{I}_N + \eta_2 \boldsymbol{H}_2 \boldsymbol{\Sigma}_{11} \boldsymbol{H}_2^\dagger\right)\right] \quad (13)$$

$$C^G_3 \triangleq \max_{\boldsymbol{\Sigma}_{11}} \log\left[\det\left(\boldsymbol{I}_{N_1} + \eta_1 \boldsymbol{H}_1 \boldsymbol{\Sigma}_{11} \boldsymbol{H}_1^\dagger\right)\right] \quad (14)$$

$$C^G_4 \triangleq \max_{\boldsymbol{\Sigma}_{22}} \log\left[\det\left(\boldsymbol{I}_N + \eta_3 \boldsymbol{H}_3 \boldsymbol{\Sigma}_{22} \boldsymbol{H}_3^\dagger \times \left(\boldsymbol{I}_N + \eta_2 \boldsymbol{H}_2 \boldsymbol{\Sigma}^*_{11} \boldsymbol{H}_2^\dagger\right)^{-1}\right)\right] \quad (15)$$

with

$$\boldsymbol{\Sigma}^*_{11} \triangleq \arg\max_{\boldsymbol{\Sigma}_{11}\succeq 0} \log\left[\det\left(\boldsymbol{I}_{N_1} + \eta_1 \boldsymbol{H}_1 \boldsymbol{\Sigma}_{11} \boldsymbol{H}_1^\dagger\right)\right].$$

We outline the procedure to compute the lower bound as follows:

Algorithm II
1. Use the water-filling technique to find C^G_d [24];
2. Use the water-filling technique to find C^G_3 and the corresponding optimal $\boldsymbol{\Sigma}^*_{11}$;
3. Substitute $\boldsymbol{\Sigma}^*_{11}$ into (15) to find C^G_4 by using the water-filling technique.

In Appendix B, we provide another lower bound by using the fact that the following rate is achievable for any given distribution $p(\boldsymbol{x}_1, \boldsymbol{x}_2)$ [4]:

$$R = \min\left(\boldsymbol{I}(\boldsymbol{X}_1; \boldsymbol{Y}_1 | \boldsymbol{X}_2),\ \boldsymbol{I}(\boldsymbol{X}_1, \boldsymbol{X}_2; \boldsymbol{Y})\right). \quad (16)$$

We note that this lower bound does not admit to a closed-form solution, but it may yield a tighter bound.

B. Proofs of Theorems 3.1 and 3.2

Proof of Theorem 3.1:

Proof: Define

$$\boldsymbol{H}_{12} \triangleq \begin{bmatrix} \sqrt{\eta_1}\boldsymbol{H}_1 \\ \sqrt{\eta_2}\boldsymbol{H}_2 \end{bmatrix}.$$

Given $\boldsymbol{X}_2 = \boldsymbol{x}_2$, we can rewrite the channel model as follows:

$$\begin{bmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y} \end{bmatrix} = \boldsymbol{H}_{12}\boldsymbol{X}_1 + \begin{bmatrix} \boldsymbol{Z}_1 \\ \boldsymbol{Z} + \sqrt{\eta_3}\boldsymbol{H}_3\boldsymbol{x}_2 \end{bmatrix}. \quad (17)$$

The sum rate of the corresponding BC channel is given by

$$\begin{aligned}
&\boldsymbol{I}(\boldsymbol{X}_1; \boldsymbol{Y}_1, \boldsymbol{Y} | \boldsymbol{X}_2) \\
&= h(\boldsymbol{Y}_1, \boldsymbol{Y} | \boldsymbol{X}_2) - h(\boldsymbol{Y}_1, \boldsymbol{Y} | \boldsymbol{X}_1, \boldsymbol{X}_2) \\
&\overset{(a)}{=} \mathbb{E}_{\boldsymbol{x}_2}\left[h(\boldsymbol{Y}_1, \boldsymbol{Y} | \boldsymbol{X}_2 = \boldsymbol{x}_2)\right] - h(\boldsymbol{Z}_1, \boldsymbol{Z}) \\
&= \mathbb{E}_{\boldsymbol{x}_2}\left[h(\boldsymbol{Y}_1, \boldsymbol{Y} | \boldsymbol{X}_2 = \boldsymbol{x}_2)\right] \\
&\quad - \log((\pi e)^{N1+N} \det(\boldsymbol{I}_{N_1+N})) \\
&\overset{(b)}{\leq} \mathbb{E}_{\boldsymbol{x}_2}\left[\log\left((\pi e)^{N1+N} \times \det\left(\operatorname{Cov}\left(\begin{bmatrix} \boldsymbol{Y} \\ \boldsymbol{Y}_1 \end{bmatrix} \middle| \boldsymbol{X}_2 = \boldsymbol{x}_2\right)\right)\right)\right] \\
&\quad - \log((\pi e)^{N1+N}) \\
&\overset{(c)}{=} \mathbb{E}_{\boldsymbol{x}_2}\left[\log\left(\det\left(\boldsymbol{I} + \boldsymbol{H}_{12}\boldsymbol{\Sigma}_{\boldsymbol{X}_1|\boldsymbol{X}_2=\boldsymbol{x}_2}\boldsymbol{H}_{12}^\dagger\right)\right)\right] \\
&\overset{(d)}{=} \log\left[\det\left(\boldsymbol{I} + \boldsymbol{H}_{12}\boldsymbol{\Sigma}_{11}^{\frac{1}{2}} \times \left(\boldsymbol{I} - \boldsymbol{\Sigma}_{11}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{22}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{12}^\dagger\boldsymbol{\Sigma}_{11}^{-\frac{1}{2}}\right)\boldsymbol{\Sigma}_{11}^{\frac{1}{2}}\boldsymbol{H}_{12}^\dagger\right)\right] \quad (18) \\
&= \log\left[\det\left(\boldsymbol{I} + \boldsymbol{H}_{12}\boldsymbol{\Sigma}_{11}^{\frac{1}{2}}\left(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^\dagger\right)\boldsymbol{\Sigma}_{11}^{\frac{1}{2}}\boldsymbol{H}_{12}^\dagger\right)\right] \\
&\overset{(e)}{\leq} \log\left[\det\left(\boldsymbol{I} + \left(1-\rho^2\right)\boldsymbol{H}_{12}\boldsymbol{\Sigma}_{11}\boldsymbol{H}_{12}^\dagger\right)\right], \quad (19)
\end{aligned}$$

where

(a) follows from the definition of conditional entropy;

(b) from the fact that circularly symmetric complex Gaussian distribution maximizes entropy [24];

(c) from the fact that

$$\operatorname{Cov}\left(\begin{bmatrix} \boldsymbol{Y} \\ \boldsymbol{Y}_1 \end{bmatrix} \middle| \boldsymbol{X}_2 = \boldsymbol{x}_2\right) = \operatorname{Cov}(\boldsymbol{H}_{12}\boldsymbol{X}_1 | \boldsymbol{X}_2 = \boldsymbol{x}_2) + \operatorname{Cov}\left(\begin{bmatrix} \boldsymbol{Z}_1 \\ \boldsymbol{Z} + \sqrt{\eta_3}\boldsymbol{H}_3\boldsymbol{x}_2 \end{bmatrix}\right)$$

which is based on that given $\boldsymbol{X}_2 = \boldsymbol{x}_2$, the vector $\begin{bmatrix} \boldsymbol{Y} \\ \boldsymbol{Y}_1 \end{bmatrix}$ is the sum of two independent circularly symmetric complex Gaussian vectors;

(d) is because that $\boldsymbol{\Sigma}_{\boldsymbol{X}_1|\boldsymbol{X}_2=\boldsymbol{x}_2}$ is the conditional covariance matrix of $\boldsymbol{X}_1$ given that $\boldsymbol{X}_2 = \boldsymbol{x}_2$, and

$$\begin{aligned}
&\boldsymbol{\Sigma}_{\boldsymbol{X}_1|\boldsymbol{X}_2=\boldsymbol{x}_2} \\
&= \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{12}^\dagger \\
&= \boldsymbol{\Sigma}_{11}^{\frac{1}{2}}\left(\boldsymbol{I} - \boldsymbol{\Sigma}_{11}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{22}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{12}^\dagger\boldsymbol{\Sigma}_{11}^{-\frac{1}{2}}\right)\boldsymbol{\Sigma}_{11}^{\frac{1}{2}}
\end{aligned}$$

which is independent of $\boldsymbol{x}_2$;

(e) from the following proof.

Applying Lemma 3.1, we have that

$$\boldsymbol{0} \preceq \boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^\dagger \preceq (1-\rho^2)\boldsymbol{I}_{M_1}.$$

Then, by [11, p. 470]

$$\begin{aligned}
\boldsymbol{H}_{12}\boldsymbol{\Sigma}_{11}^{\frac{1}{2}}\left(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^\dagger\right)\boldsymbol{\Sigma}_{11}^{\frac{1}{2}}\boldsymbol{H}_{12}^\dagger &\preceq \boldsymbol{H}_{12}\boldsymbol{\Sigma}_{11}^{\frac{1}{2}}\left(1-\rho^2\right)\boldsymbol{I}_{M_1}\boldsymbol{\Sigma}_{11}^{\frac{1}{2}}\boldsymbol{H}_{12}^\dagger \\
&= \left(1-\rho^2\right)\boldsymbol{H}_{12}\boldsymbol{\Sigma}_{11}\boldsymbol{H}_{12}^\dagger.
\end{aligned}$$

It follows that

$$\boldsymbol{I} + \boldsymbol{H}_{12}\boldsymbol{\Sigma}_{11}^{\frac{1}{2}}\left(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^\dagger\right)\boldsymbol{\Sigma}_{11}^{\frac{1}{2}}\boldsymbol{H}_{12}^\dagger \preceq \boldsymbol{I} + \left(1-\rho^2\right)\boldsymbol{H}_{12}\boldsymbol{\Sigma}_{11}\boldsymbol{H}_{12}^\dagger.$$

Observing that both sides in the above expression are positive definite, we have that

$$\begin{aligned}
0 &< \det\left(\boldsymbol{I} + \boldsymbol{H}_{12}\boldsymbol{\Sigma}_{11}^{\frac{1}{2}}\left(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^\dagger\right)\boldsymbol{\Sigma}_{11}^{\frac{1}{2}}\boldsymbol{H}_{12}^\dagger\right) \\
&\leq \det\left(\boldsymbol{I} + \left(1-\rho^2\right)\boldsymbol{H}_{12}\boldsymbol{\Sigma}_{11}\boldsymbol{H}_{12}^\dagger\right).
\end{aligned}$$

We conclude that

$$\log\left[\det\left(\boldsymbol{I}+\boldsymbol{H}_{12}\boldsymbol{\Sigma}_{11}^{\frac{1}{2}}\left(\boldsymbol{I}-\boldsymbol{A}\boldsymbol{A}^{\dagger}\right)\boldsymbol{\Sigma}_{11}^{\frac{1}{2}}\boldsymbol{H}_{12}^{\dagger}\right)\right] \leq \log\left[\det\left(\boldsymbol{I}+(1-\rho^2)\,\boldsymbol{H}_{12}\boldsymbol{\Sigma}_{11}\boldsymbol{H}_{12}^{\dagger}\right)\right].$$

In summary, we have shown that

$$\boldsymbol{I}(\boldsymbol{X}_1;\boldsymbol{Y}_1,\boldsymbol{Y}|\boldsymbol{X}_2) \leq \log\left[\det\left(\boldsymbol{I}+\left(1-\rho^2\right)\boldsymbol{H}_{12}\boldsymbol{\Sigma}_{11}\boldsymbol{H}_{12}^{\dagger}\right)\right]. \tag{20}$$

Next, we turn the attention to the sum rate of the MAC part. Observe (21) and (22) at the bottom of the page, where

(a) from the fact that the circularly symmetric complex Gaussian distribution maximizes entropy [24] as mentioned before;

(b) can be shown as follows. First, by Lemma 3.2, we have that for $\forall\, a > 0$

$$\sqrt{\eta_2\eta_3}\boldsymbol{H}_2\boldsymbol{\Sigma}_{12}\boldsymbol{H}_3^{\dagger}+\sqrt{\eta_2\eta_3}\left(\boldsymbol{H}_2\boldsymbol{\Sigma}_{12}\boldsymbol{H}_3^{\dagger}\right)^{\dagger} \preceq \frac{\rho^2}{a}\eta_2\boldsymbol{H}_2\boldsymbol{\Sigma}_{11}\boldsymbol{H}_2^{\dagger}+a\eta_3\boldsymbol{H}_3\boldsymbol{\Sigma}_{22}\boldsymbol{H}_3^{\dagger}.$$

Next, following the same procedures as above, it is easy to show that

$$\begin{aligned}\log&\left[\det\left(\boldsymbol{I}+\eta_2\boldsymbol{H}_2\boldsymbol{\Sigma}_{11}\boldsymbol{H}_2^{\dagger}+\eta_3\boldsymbol{H}_3\boldsymbol{\Sigma}_{22}\boldsymbol{H}_3^{\dagger}\right.\right.\\&\left.\left.+\sqrt{\eta_2\eta_3}\boldsymbol{H}_2\boldsymbol{\Sigma}_{12}\boldsymbol{H}_3^{\dagger}+\sqrt{\eta_2\eta_3}\boldsymbol{H}_3\boldsymbol{\Sigma}_{21}\boldsymbol{H}_2^{\dagger}\right)\right]\\&\leq\log\left[\det\left(\boldsymbol{I}+\eta_2\boldsymbol{H}_2\boldsymbol{\Sigma}_{11}\boldsymbol{H}_2^{\dagger}+\eta_3\boldsymbol{H}_3\boldsymbol{\Sigma}_{22}\boldsymbol{H}_3^{\dagger}\right.\right.\\&\left.\left.\quad+\frac{\rho^2}{a}\sqrt{\eta_2\eta_3}\boldsymbol{H}_2\boldsymbol{\Sigma}_{11}\boldsymbol{H}_2^{\dagger}+a\sqrt{\eta_2\eta_3}\boldsymbol{H}_3\boldsymbol{\Sigma}_{22}\boldsymbol{H}_3^{\dagger}\right)\right].\end{aligned}$$

Finally, taking the infimum on both sides yields (22). □

Proof of Theorem 3.2:

Proof: If there is no relay node, the channel is a point-to-point MIMO channel, and the corresponding channel capacity is given by

$$C_d^G \triangleq \max_{\boldsymbol{\Sigma}_{11}}\log\left[\det\left(\boldsymbol{I}_N+\eta_2\boldsymbol{H}_2\boldsymbol{\Sigma}_{11}\boldsymbol{H}_2^{\dagger}\right)\right]. \tag{23}$$

Consider the case where the destination node treats the signal from the source node as noise. In this case, the source node optimizes its transmission only for the source–relay link, and the relay node optimizes the transmission corresponding to the relay–destination link. That is, the channel boils down to a cascaded channel. The following information rate is achievable for the source–relay link:

$$C_3^G = \max_{\boldsymbol{\Sigma}_{11}}\log\left[\det\left(\boldsymbol{I}_{N_1}+\eta_1\boldsymbol{H}_1\boldsymbol{\Sigma}_{11}\boldsymbol{H}_1^{\dagger}\right)\right] \tag{24}$$

and the corresponding optimal covariance matrix is $\boldsymbol{\Sigma}_{11}^*$. For the relay–destination link, the received signal at the destination is distorted by both noise and the signal from the source node. Since the (optimal) signal from the source is also Gaussian, with covariance matrix $\boldsymbol{\Sigma}_{11}^*$, the covariance matrix for noise plus the source signal is $\boldsymbol{I}_N + \eta_2\boldsymbol{H}_2\boldsymbol{\Sigma}_{11}^*\boldsymbol{H}_2^{\dagger}$. Therefore, the achievable information rate for the relay–destination link is given by

$$\begin{aligned}C_4^G &= \max_{\boldsymbol{\Sigma}_{22}}\log\left[\det\left(\left(\boldsymbol{I}_N+\eta_2\boldsymbol{H}_2\boldsymbol{\Sigma}_{11}^*\boldsymbol{H}_2^{\dagger}+\eta_3\boldsymbol{H}_3\boldsymbol{\Sigma}_{22}\boldsymbol{H}_3^{\dagger}\right)\right.\right.\\&\qquad\left.\left.\times\left(\boldsymbol{I}_N+\eta_2\boldsymbol{H}_2\boldsymbol{\Sigma}_{11}^*\boldsymbol{H}_2^{\dagger}\right)^{-1}\right)\right]\\&=\max_{\boldsymbol{\Sigma}_{22}}\log\left[\det\left(\boldsymbol{I}_N+\eta_3\boldsymbol{H}_3\boldsymbol{\Sigma}_{22}\boldsymbol{H}_3^{\dagger}\right.\right.\\&\qquad\left.\left.\times\left(\boldsymbol{I}_N+\eta_2\boldsymbol{H}_2\boldsymbol{\Sigma}_{11}^*\boldsymbol{H}_2^{\dagger}\right)^{-1}\right)\right].\end{aligned} \tag{25}$$

□

The lower bound follows by combining (23) with (24) and (25).

$$\begin{aligned}\boldsymbol{I}(\boldsymbol{X}_1,\boldsymbol{X}_2;\boldsymbol{Y}) &= h(\boldsymbol{Y})-h(\boldsymbol{Y}|\boldsymbol{X}_1,\boldsymbol{X}_2)\\&=h\left([\sqrt{\eta_2}\boldsymbol{H}_2,\sqrt{\eta_3}\boldsymbol{H}_3]\begin{bmatrix}\boldsymbol{X}_1\\\boldsymbol{X}_2\end{bmatrix}+\boldsymbol{Z}\right)-h(\boldsymbol{Z})\\&=h\left([\sqrt{\eta_2}\boldsymbol{H}_2,\sqrt{\eta_3}\boldsymbol{H}_3]\begin{bmatrix}\boldsymbol{X}_1\\\boldsymbol{X}_2\end{bmatrix}+\boldsymbol{Z}\right)-\log\left[(\pi e)^N\det(\boldsymbol{I}_N)\right]\\&\overset{(a)}{\leq}\log\left[(\pi e)^N\det\left(\mathrm{Cov}\left([\sqrt{\eta_2}\boldsymbol{H}_2,\sqrt{\eta_3}\boldsymbol{H}_3]\begin{bmatrix}\boldsymbol{X}_1\\\boldsymbol{X}_2\end{bmatrix}+\boldsymbol{Z}\right)\right)\right]-\log((\pi e)^N)\\&=\log\left[\det\left(\boldsymbol{I}+[\sqrt{\eta_2}\boldsymbol{H}_2,\sqrt{\eta_3}\boldsymbol{H}_3]\begin{bmatrix}\boldsymbol{\Sigma}_{11}&\boldsymbol{\Sigma}_{12}\\\boldsymbol{\Sigma}_{21}&\boldsymbol{\Sigma}_{22}\end{bmatrix}\begin{bmatrix}\sqrt{\eta_2}\boldsymbol{H}_2^{\dagger}\\\sqrt{\eta_3}\boldsymbol{H}_3^{\dagger}\end{bmatrix}\right)\right]\\&=\log\left[\det\left(\boldsymbol{I}+\eta_2\boldsymbol{H}_2\boldsymbol{\Sigma}_{11}\boldsymbol{H}_2^{\dagger}+\eta_3\boldsymbol{H}_3\boldsymbol{\Sigma}_{22}\boldsymbol{H}_3^{\dagger}+\sqrt{\eta_2\eta_3}\boldsymbol{H}_2\boldsymbol{\Sigma}_{12}\boldsymbol{H}_3^{\dagger}+\sqrt{\eta_2\eta_3}\boldsymbol{H}_3\boldsymbol{\Sigma}_{21}\boldsymbol{H}_2^{\dagger}\right)\right] &(21)\\&\overset{(b)}{\leq}\inf_{a>0}\log\left[\det\left(\boldsymbol{I}+\eta_2\boldsymbol{H}_2\boldsymbol{\Sigma}_{11}\boldsymbol{H}_2^{\dagger}+\eta_3\boldsymbol{H}_3\boldsymbol{\Sigma}_{22}\boldsymbol{H}_3^{\dagger}+\frac{\rho^2}{a}\sqrt{\eta_2\eta_3}\boldsymbol{H}_2\boldsymbol{\Sigma}_{11}\boldsymbol{H}_2^{\dagger}+a\sqrt{\eta_2\eta_3}\boldsymbol{H}_3\boldsymbol{\Sigma}_{22}\boldsymbol{H}_3^{\dagger}\right)\right]\\&=\inf_{a>0}\log\left[\det\left(\boldsymbol{I}+\left(\eta_2+\frac{\rho^2}{a}\sqrt{\eta_2\eta_3}\right)\boldsymbol{H}_2\boldsymbol{\Sigma}_{11}\boldsymbol{H}_2^{\dagger}+(\eta_3+a\sqrt{\eta_2\eta_3})\boldsymbol{H}_3\boldsymbol{\Sigma}_{22}\boldsymbol{H}_3^{\dagger}\right)\right] &(22)\end{aligned}$$

IV. Capacity Bounds: The Rayleigh Fading Case

Now consider channel models where all the channel gain matrices are random, and suppose that the channel gains are known at the corresponding receivers only. In this scenario, we study the ergodic capacity of the MIMO relay channel with receiver CSI only. Simply put, the ergodic capacity is the highest achievable data rate by coding the transmission symbols over infinitely many blocks [34, p. 11].

Theorem 4.1: **[Rayleigh Fading Case]** An upper bound on the ergodic capacity of the MIMO relay channel is given by

$$C^R \le C^R_{\text{upper}} = \min\left(C^R_1,\, C^R_2\right) \tag{26}$$

with

$$C^R_1 \triangleq \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I}_{M_1} + \eta_1 \boldsymbol{H}_1^\dagger \boldsymbol{H}_1 + \eta_2 \boldsymbol{H}_2^\dagger \boldsymbol{H}_2\right)\right] \tag{27}$$

$$C^R_2 \triangleq \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I}_N + \eta_2 \boldsymbol{H}_2 \boldsymbol{H}_2^\dagger + \eta_3 \boldsymbol{H}_3 \boldsymbol{H}_3^\dagger\right)\right] \tag{28}$$

where the expectations are taken with respect to channel matrices $\boldsymbol{H}_1$, $\boldsymbol{H}_2$, and $\boldsymbol{H}_3$.

Proof: Because of fading, the channel matrices are now random. Then it follows that

$$\begin{aligned} C^R &\le C^R_{\text{upper}} \\ &= \max_{p(\boldsymbol{x}_1,\boldsymbol{x}_2)} \min(\mathbb{E}_{\boldsymbol{H}} \boldsymbol{I}(\boldsymbol{X}_1;\boldsymbol{Y},\boldsymbol{Y}_1|\boldsymbol{X}_2,\boldsymbol{H}_1,\boldsymbol{H}_2), \\ &\qquad \mathbb{E}_{\boldsymbol{H}} \boldsymbol{I}(\boldsymbol{X}_1,\boldsymbol{X}_2;\boldsymbol{Y}|\boldsymbol{H}_2,\boldsymbol{H}_3)) \end{aligned} \tag{29}$$

where the expectations are taken with respect to corresponding channel coefficients.

In the proof for Theorem 3.1, we have shown that

$$C^G_{\text{upper}} = \max_{\boldsymbol{\Sigma}_{11},\boldsymbol{\Sigma}_{22},\boldsymbol{\Sigma}_{12}} \min\left(C_1', C_2'\right) \tag{30}$$

where

$$C_1' \triangleq \log\left[\det\left(\boldsymbol{I} + \boldsymbol{H}_{12}\left(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{12}^\dagger\right)\boldsymbol{H}_{12}^\dagger\right)\right] \tag{31}$$

$$C_2' \triangleq \log\left[\det\left(\boldsymbol{I} + \boldsymbol{H}_{23}\begin{bmatrix}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{bmatrix}\boldsymbol{H}_{23}^\dagger\right)\right] \tag{32}$$

with $\boldsymbol{H}_{23} \triangleq \left[\sqrt{\eta_2}\boldsymbol{H}_2, \sqrt{\eta_3}\boldsymbol{H}_3\right]$.

Observe the power constraints on input signals

$$\mathbb{E}\left(\boldsymbol{X}_1^\dagger \boldsymbol{X}_1\right) = \text{tr}(\boldsymbol{\Sigma}_{11}) \le M_1$$

and

$$\mathbb{E}\left(\boldsymbol{X}_2^\dagger \boldsymbol{X}_2\right) = \text{tr}(\boldsymbol{\Sigma}_{22}) \le M_2.$$

Then it follows that

$$\text{tr}\left(\begin{bmatrix}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{bmatrix}\right) = \text{tr}(\boldsymbol{\Sigma}_{11}) + \text{tr}(\boldsymbol{\Sigma}_{22}) \le M_1 + M_2.$$

Along the same lines as in [13], we conclude that the optimal signal covariance matrices which maximize $\mathbb{E}_{\boldsymbol{H}}(C_2')$ are identity matrices, i.e., $\boldsymbol{\Sigma}_{11} = \boldsymbol{I}_{M_1}$, $\boldsymbol{\Sigma}_{22} = \boldsymbol{I}_{M_2}$, and $\boldsymbol{\Sigma}_{12} = \boldsymbol{0}$. Therefore,

$$\begin{aligned} \mathbb{E}_{\boldsymbol{H}}(C_2') &\le \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I} + \boldsymbol{H}_{23}\boldsymbol{H}_{23}^\dagger\right)\right] \\ &= \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I}_N + \eta_2 \boldsymbol{H}_2 \boldsymbol{H}_2^\dagger + \eta_3 \boldsymbol{H}_3 \boldsymbol{H}_3^\dagger\right)\right]. \end{aligned} \tag{33}$$

Note that $\boldsymbol{H}_{12}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{12}^\dagger\boldsymbol{H}_{12}^\dagger$ is nonnegative with probability 1 [9, Theorem 3.2]. Along the same line of the proof of Theorem 3.1, we have that

$$\mathbb{E}_{\boldsymbol{H}}(C_1') \le \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I} + \boldsymbol{H}_{12}\boldsymbol{\Sigma}_{11}\boldsymbol{H}_{12}^\dagger\right)\right].$$

Furthermore, by [24, Theorem 1], $\boldsymbol{\Sigma}_{11} = \boldsymbol{I}_{M_1}$ can also maximize the RHS of the preceding equation. We conclude that the covariance matrices that maximize $\mathbb{E}_{\boldsymbol{H}}(C_2')$ can also maximize $\mathbb{E}_{\boldsymbol{H}}(C_1')$, i.e.,

$$\begin{aligned} \mathbb{E}_{\boldsymbol{H}}(C_1') &\le \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I} + \boldsymbol{H}_{12}\boldsymbol{H}_{12}^\dagger\right)\right] \\ &= \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I}_N + \eta_1 \boldsymbol{H}_1^\dagger \boldsymbol{H}_1 + \eta_2 \boldsymbol{H}_2^\dagger \boldsymbol{H}_2\right)\right]. \end{aligned} \tag{34}$$

In a nutshell, the mutual information rates in (29) are maximized by choosing $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ to be independent circularly symmetric complex Gaussian vectors with $\boldsymbol{\Sigma}_{11} = \boldsymbol{I}_{M_1}$, $\boldsymbol{\Sigma}_{22} = \boldsymbol{I}_{M_2}$, and $\boldsymbol{\Sigma}_{12} = \boldsymbol{0}$. □

In what follows, we present a lower bound on the ergodic capacity.

Theorem 4.2: **[Rayleigh Fading Case]** A lower bound on the ergodic capacity of the MIMO relay channel is given by

$$C^R \ge C^R_{\text{lower}} = \max\left(C^R_d,\, \min\left(C^R_3,\, C^R_2\right)\right) \tag{35}$$

with

$$C^R_d \triangleq \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I}_N + \eta_2 \boldsymbol{H}_2 \boldsymbol{H}_2^\dagger\right)\right] \tag{36}$$

$$C^R_3 \triangleq \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I}_{N_1} + \eta_1 \boldsymbol{H}_1 \boldsymbol{H}_1^\dagger\right)\right] \tag{37}$$

where the expectations are taken with respect to corresponding channel matrices.

Proof: Based on [4, Sec. VI], the following rate is achievable by using block Markov coding:

$$R = \max_{p(\boldsymbol{x}_1,\boldsymbol{x}_2)} \min(\boldsymbol{I}(\boldsymbol{X}_1;\boldsymbol{Y}_1|\boldsymbol{X}_2), \boldsymbol{I}(\boldsymbol{X}_1,\boldsymbol{X}_2;\boldsymbol{Y})). \tag{38}$$

Since the receivers have full CSI, it follows that

$$\boldsymbol{I}(\boldsymbol{X}_1;\boldsymbol{Y}_1|\boldsymbol{X}_2) = \mathbb{E}_{\boldsymbol{H}}[\boldsymbol{I}(\boldsymbol{X}_1;\boldsymbol{Y}_1|\boldsymbol{X}_2,\boldsymbol{H}_1)] \tag{39}$$

$$\boldsymbol{I}(\boldsymbol{X}_1,\boldsymbol{X}_2;\boldsymbol{Y}) = \mathbb{E}_{\boldsymbol{H}}[\boldsymbol{I}(\boldsymbol{X}_1,\boldsymbol{X}_2;\boldsymbol{Y}|\boldsymbol{H}_2,\boldsymbol{H}_3)] \tag{40}$$

where the expectations are taken with respect to the corresponding channel matrices. Note that

$$\begin{aligned} &\boldsymbol{I}(\boldsymbol{X}_1;\boldsymbol{Y}_1|\boldsymbol{X}_2,\boldsymbol{H}_1) \\ &\quad \le \log\left[\det\left(\boldsymbol{I} + \boldsymbol{H}_1\left(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{12}^\dagger\right)\boldsymbol{H}_1^\dagger\right)\right]. \end{aligned}$$

Let $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ be independent circularly symmetric complex Gaussian random vectors with $\boldsymbol{\Sigma}_{11} = \boldsymbol{I}_{M_1}$, $\boldsymbol{\Sigma}_{22} = \boldsymbol{I}_{M_2}$, and $\boldsymbol{\Sigma}_{12} = \boldsymbol{0}$. Then, the lower bound in (35) follows along the same line of the proof of Theorem 4.1. □

Remarks: Interestingly, Theorems 4.1 and 4.2 reveal that in a Rayleigh-fading channel, the corresponding codebooks at

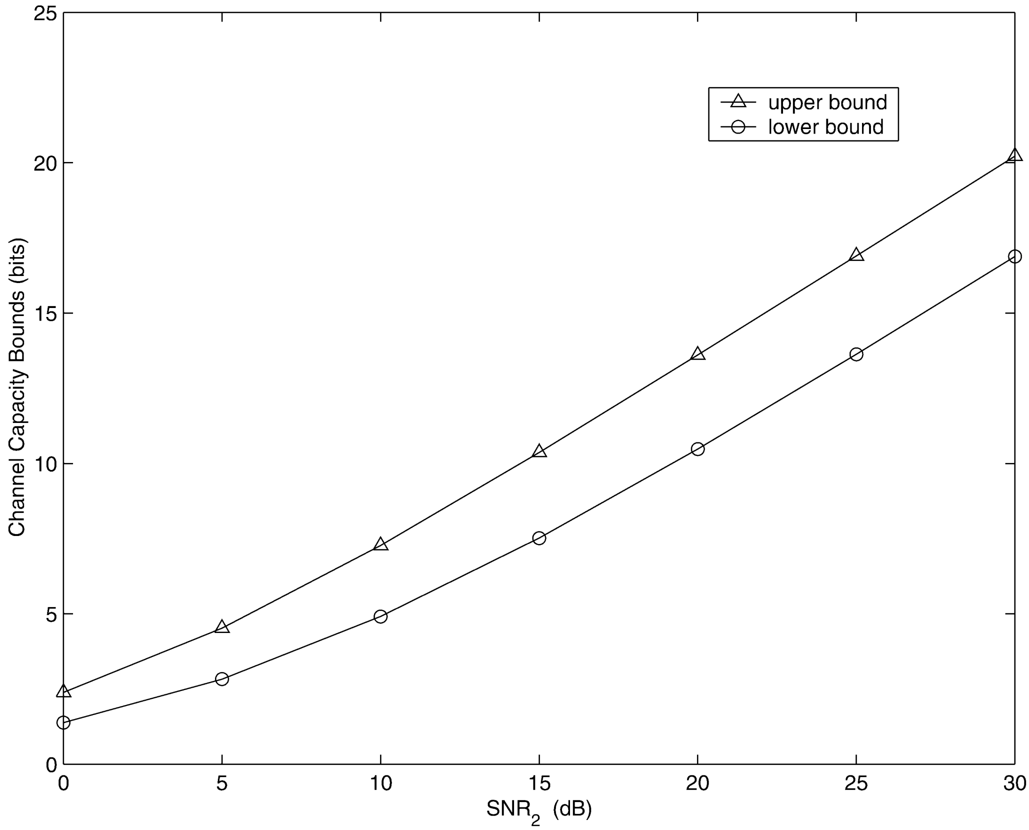

Fig. 3. Capacity bounds versus SNR_2 for the case $\eta_1 = \eta_2 = \eta_3$.

the source node and the relay node are independent, i.e., $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent. (As shown later, the upper bound and the lower bound can "converge" under certain conditions, indicating that the ergodic capacity of the MIMO relay channel can be characterized exactly.) In contrast, $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are correlated for the fixed channel cases. Indeed, the capacity of the MAC part is achieved when the source node and the relay node have "complete" cooperation for the fixed channel case [4]. Our intuition for this finding is as follows. Since we consider the relay channel with receiver CSI only, the transmitters have no knowledge about the channel realizations. As a result, the optimal codebooks at the source node and the relay node are independent, due to the channel uncertainty at the transmitters. Intuitively speaking, it is the uncertainty (randomness) of $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ at the transmitters that makes $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ independent.

Recall that in a single-user MIMO channel with receiver CSI only, the capacity is achieved when the power allocation across transmit antennas is equal and the signals are independent [24, Theorem1]. If we treated the relay node and the source node as an antenna-clustering transmitter [7], the optimal signaling would indicate independent signals across transmit antennas.

It is clear that the communications between the source node and the destination node can be improved by using relaying (see, e.g., Case III in Section IV-A). From (27) and (28), the capacity gain for the MAC part is due to the multiple-access gain; and the capacity gain for the BC part originates from the broadcast gain. Needless to say, a key to reap the capacity gains is to develop coding strategies for the cooperative MAC and the cooperative BC therein.

A. Numerical Examples

We now illustrate via numerical examples the bounds in Theorems 4.1 and 4.2. For the sake of clarity, we consider the case where the numbers of antennas at all the transmitters and receivers are equal (e.g., in *ad hoc* networks, all the nodes are equipped with identical RF devices). We study the upper bound and the lower bound with different SNR parameters (namely, η_1, η_2, and η_3). The SNR parameters play a key role in the upper bound and the lower bound. In what follows, we study three cases with different SNR parameters. The number of the antennas is assumed to be two in all cases, and $\eta_2 = \mathrm{SNR}_2/2$, with SNR_2 being the SNR for the direct link.

Case I: In this case, $\eta_1 = \eta_2 = \eta_3$; this models the scenario where the source node, the relay node, and the destination node are separated by equal distances. Fig. 3 depicts the upper bound and the lower bound.

Case II: In this case, $\eta_1 = \eta_2$ and $\eta_3 = 10\eta_2$, which "captures" that the relay node is closer to the destination node than to the source node. Comparing Fig. 4 with Fig. 3, we observe that the upper bound and the lower bound for Case I and Case II are the same. We will elaborate further on this in Section V (see Lemma 5.3).

Case III: In this case, $\eta_2 = \eta_3$ and $\eta_1 = 10\eta_2$, which "captures" a scenario that the relay node is closer to the source node than to the destination node. Surprisingly, as shown in Fig. 5, the upper bound and the lower bound "converge." That is to say, the ergodic capacity of the MIMO relay channel over Rayleigh

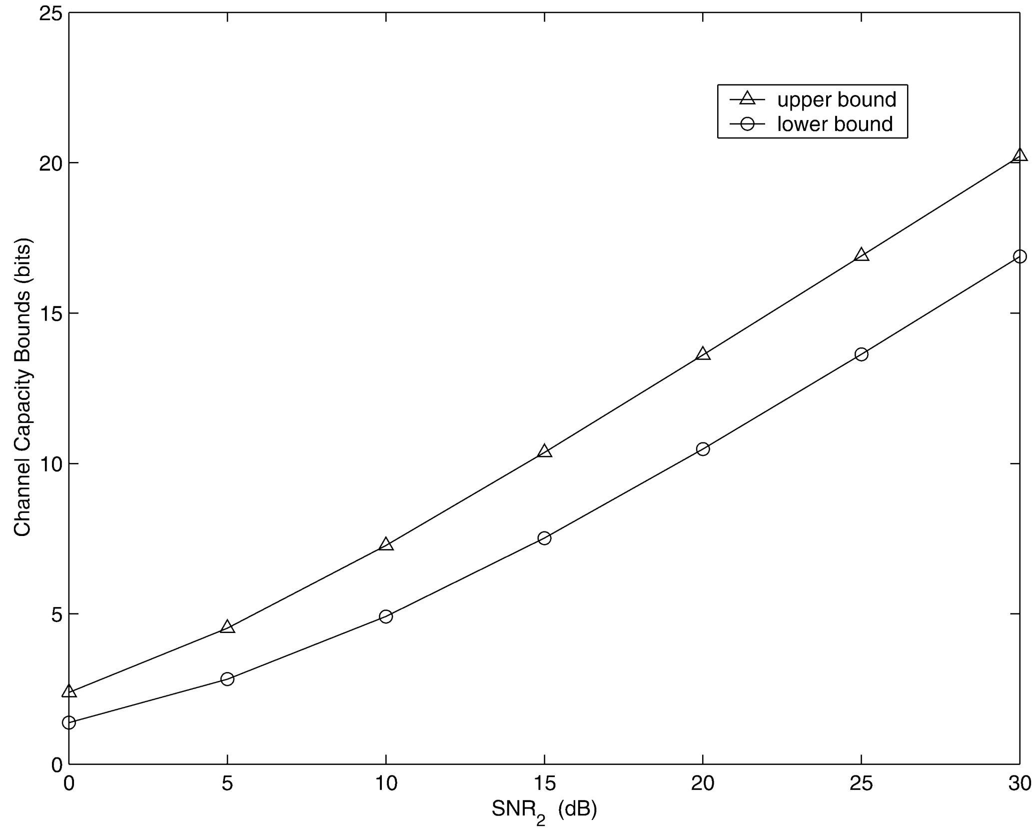

Fig. 4. Capacity bounds versus SNR_2 for the case $\eta_1 = \eta_2$ and $\eta_3 = 10\eta_1$.

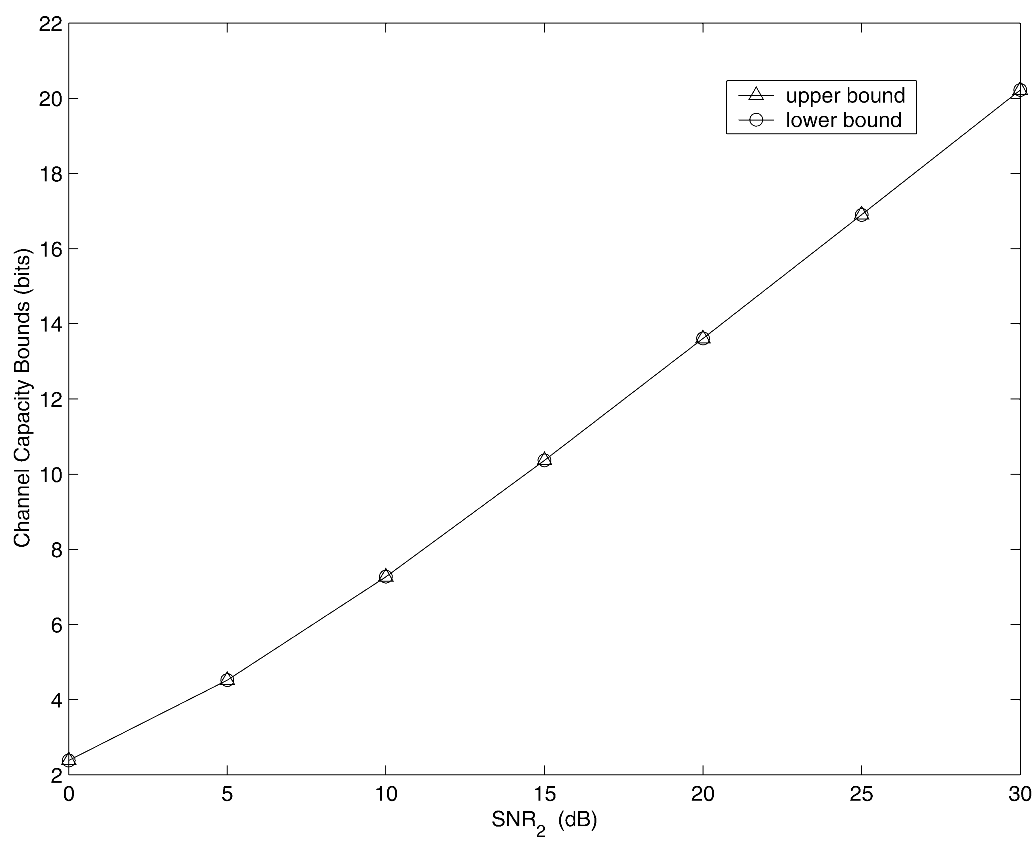

Fig. 5. Capacity bounds versus SNR_2 for the case $\eta_2 = \eta_3$ and $\eta_1 = 10\eta_2$.

fading can be characterized under this SNR condition. We will discuss in Section V sufficient conditions under which the ergodic capacity can be achieved.

V. Discussions On Capacity Achievability

The Gaussian MIMO relay channel with fixed channel gains can be viewed as a vector generalization of the classical Gaussian relay channel in [4], and is not degraded in general. Characterizing the corresponding capacity remains open. In the following, we turn our attention to the fading case. Specifically, we investigate in Section V-A sufficient conditions that give exact characterization of the ergodic capacity. Since the ergodic capacity involves expectations with respect to random matrices and does not admit an "explicit" expression, we study in Section V-B the high-SNR regime and use approximations to identify SNR conditions for achieving the capacity; in Section V-C, we examine the scale case, for which we derive explicit conditions for capacity achievability and the explicit capacity expression.

A. Regularity Conditions for Capacity Achievability

In Section IV, we have presented a lower bound and an upper bound on the ergodic capacity of the MIMO relay channel and also provided numerical examples for different SNR cases. Interestingly, the upper bound and the lower bound in Fig. 5 "converge," which indicates that under certain regularity conditions, the ergodic capacity of the MIMO relay channel can be characterized exactly. Indeed, we observe that in (26) and (35), there is a common term C_2^R. If C_2^R is smaller than both C_1^R and C_3^R, the ergodic capacity of the MIMO relay channel is given by C_2^R. We state this important observation in the following proposition.

Proposition 5.1: If $C_1^R \geq C_2^R$ and $C_3^R \geq C_2^R$, then the ergodic capacity of the MIMO relay channel is given by C_2^R.

In what follows, we study the conditions in Proposition 5.1 in terms of SNR parameters. For tractability, we assume that all the numbers of antennas are equal, i.e., $M_1 = M_2 = N_1 = N$ (for simplicity, we use N to denote the number of the antennas). We first need the following lemma.

Lemma 5.1: The upper bound on the ergodic capacity of the MIMO relay channel is C_2^R, if $\eta_1 \geq \eta_3$.

Proof: Under the assumption on equal numbers of antennas, it follows that $\boldsymbol{H}_i^\dagger \boldsymbol{H}_i$ and $\boldsymbol{H}_i \boldsymbol{H}_i^\dagger$ have identical probability distributions, for $i = 1, 2, 3$. Therefore,

$$\begin{aligned} C_1^R &= \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I}_N + \eta_1 \boldsymbol{H}_1 \boldsymbol{H}_1^\dagger + \eta_2 \boldsymbol{H}_2 \boldsymbol{H}_2^\dagger\right)\right] \\ &= \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I}_N + \eta_1 \boldsymbol{H}_3 \boldsymbol{H}_3^\dagger + \eta_2 \boldsymbol{H}_2 \boldsymbol{H}_2^\dagger\right)\right]. \end{aligned}$$

The second equality follows the fact that $\boldsymbol{H}_1 \boldsymbol{H}_1^\dagger$, $\boldsymbol{H}_2 \boldsymbol{H}_2^\dagger$, and $\boldsymbol{H}_3 \boldsymbol{H}_3^\dagger$ are i.i.d. Wishart matrices, and accordingly $(\boldsymbol{H}_1 \boldsymbol{H}_1^\dagger, \boldsymbol{H}_2 \boldsymbol{H}_2^\dagger)$ and $(\boldsymbol{H}_3 \boldsymbol{H}_3^\dagger, \boldsymbol{H}_2 \boldsymbol{H}_2^\dagger)$ follow identical probability laws. Furthermore, since C_1^R increases monotonically with η_1, we have that $C_1^R \geq C_2^R$ if $\eta_1 \geq \eta_3$. □

Lemma 5.1 gives the conditions under which the upper bound is C_2^R. It remains to examine when C_2^R would meet the lower bound, that is, $C_2^R \leq C_3^R$ (it is easy to show that C_2^R is always greater than C_d^R), and thus, the ergodic capacity can be characterized. In general, it is nontrivial to determine in terms of η_1, η_2, and η_3 if $C_2^R \leq C_3^R$ (except the scalar case), because the ergodic capacity expressions involve expectations with respect to random matrices. In light of this fact, we first find an upper bound on C_2^R and compare it with C_3^R. The following lemma provides an upper bound on C_2^R.

Lemma 5.2: For any $\eta_2 \geq 0$ and $\eta_3 \geq 0$

$$C_2^R \leq \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I}_N + \frac{\eta_2 + \eta_3}{2}\left(\boldsymbol{H}_2 \boldsymbol{H}_2^\dagger + \boldsymbol{H}_3 \boldsymbol{H}_3^\dagger\right)\right)\right].$$

Proof: First, we rewrite C_2^R as

$$C_2^R = \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I}_N + [\boldsymbol{H}_2, \boldsymbol{H}_3]\begin{bmatrix} \eta_2 \boldsymbol{I}_N & \boldsymbol{0} \\ \boldsymbol{0} & \eta_3 \boldsymbol{I}_N \end{bmatrix}\begin{bmatrix} \boldsymbol{H}_2^\dagger \\ \boldsymbol{H}_3^\dagger \end{bmatrix}\right)\right].$$

Along the same line of the proof in [24, Theorem 1], it can be shown that if the total power is kept constant, i.i.d. input signals with the equal power allocation can maximize the mutual information. It then follows that

$$\begin{aligned} C_2^R &\leq \mathbb{E}_{\boldsymbol{H}} \log\Bigg[\det\Bigg(\boldsymbol{I}_N + [\boldsymbol{H}_2, \boldsymbol{H}_3] \\ &\qquad \times \begin{bmatrix} \frac{\eta_2 N + \eta_3 N}{2N} \boldsymbol{I}_N & \boldsymbol{0} \\ \boldsymbol{0} & \frac{\eta_2 N + \eta_3 N}{2N} \boldsymbol{I}_N \end{bmatrix}\begin{bmatrix} \boldsymbol{H}_2^\dagger \\ \boldsymbol{H}_3^\dagger \end{bmatrix}\Bigg)\Bigg] \\ &= \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I}_N + \frac{\eta_2 + \eta_3}{2}[\boldsymbol{H}_2, \boldsymbol{H}_3]\begin{bmatrix} \boldsymbol{H}_2^\dagger \\ \boldsymbol{H}_3^\dagger \end{bmatrix}\right)\right]. \end{aligned}$$

It is clear that the equality can be achieved if $\eta_2 = \eta_3$. □

In Section V-B, we will derive sufficient conditions for capacity achievability by combining Proposition 5.1 with Lemmas 5.1 and 5.2, building on which we elaborate further on the numerical results exhibited in Case III in Section IV-A.

Next, we present in Lemma 5.3 the conditions under which the upper bound and the lower bound diverge. This sheds light on the existence of the gap between the upper bound and the lower bound in Figs. 3 and 4.

Lemma 5.3: If $\eta_1 = \eta_2$ and $\eta_1 \leq \eta_3$, then the upper bound on the ergodic capacity of the MIMO relay channel is C_1^R, and the lower bound is C_3^R.

Proof: The first statement directly follows Lemma 5.1, and it remains to show the second one. We have that

$$\begin{aligned} C_2^R &= \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I}_N + \eta_2 \boldsymbol{H}_2 \boldsymbol{H}_2^\dagger + \eta_3 \boldsymbol{H}_3 \boldsymbol{H}_3^\dagger\right)\right] \\ &\overset{(a)}{\geq} \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I}_N + \eta_2 \boldsymbol{H}_2 \boldsymbol{H}_2^\dagger\right)\right] \\ &\overset{(b)}{=} \mathbb{E}_{\boldsymbol{H}} \log\left[\det\left(\boldsymbol{I}_N + \eta_1 \boldsymbol{H}_1 \boldsymbol{H}_1^\dagger\right)\right] \end{aligned}$$

where (a) follows from the fact that $\boldsymbol{H}_3 \boldsymbol{H}_3^\dagger \succ \boldsymbol{0}$ with probability 1 [9, Theorem 3.2]; and (b) follows from the facts that $\eta_1 = \eta_2$, and $\boldsymbol{H}_1 \boldsymbol{H}_1^\dagger$ and $\boldsymbol{H}_2 \boldsymbol{H}_2^\dagger$ are i.i.d. random matrices, as mentioned earlier. □

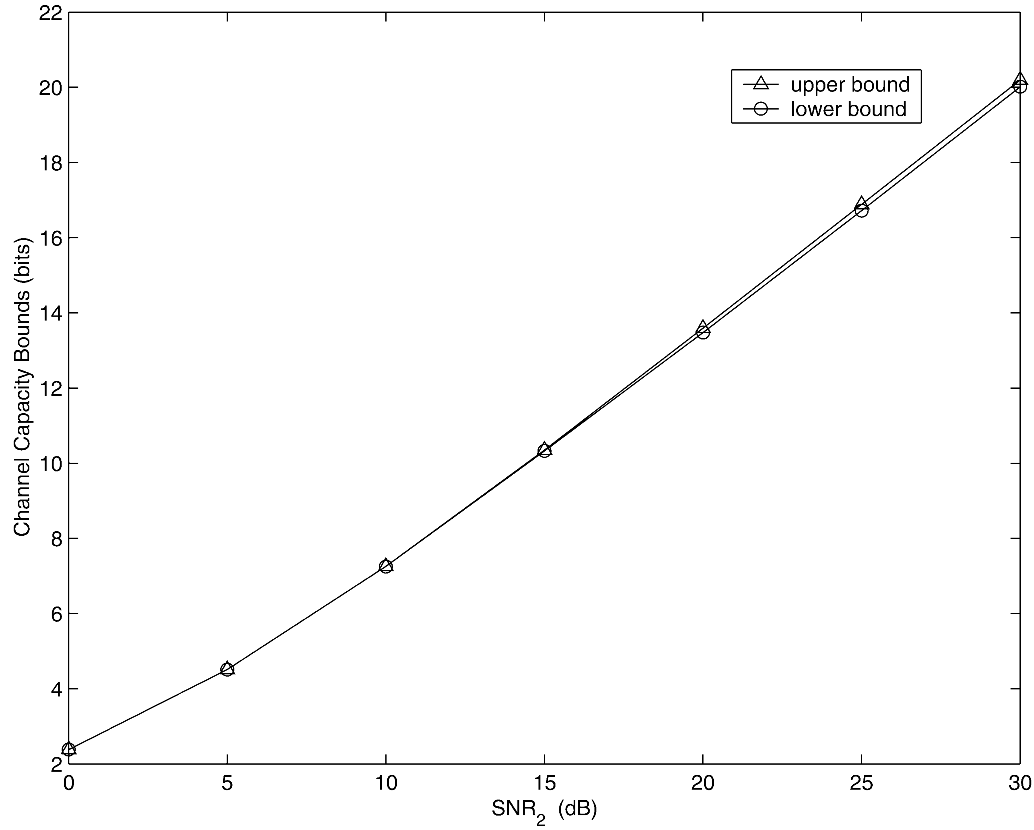

Fig. 6. Capacity bounds versus SNR_2 for the case of $\eta_2 = \eta_3$ and $\eta_1 = 3\eta_2$.

Next, we examine two cases where the capacity-achieving conditions can be expressed in explicit form.

B. Achievability of Ergodic Capacity: The High SNR Regime

In the high-SNR regime [19, Proposition 2], i.e., η_1, η_2, and η_3 are large, C_3^R can be approximated as

$$C_3^R \approx N\log 2(\eta_1) + \frac{1}{\ln 2}\left(\sum_{j=1}^{N}\sum_{p=1}^{N-j}\frac{1}{p} - \gamma N\right) \tag{41}$$

where $\gamma \approx 0.57721566$ is Euler's constant. The same approximation can be applied to the upper bound on C_2^R in Lemma 5.2; i.e.,

$$C_2^R \leq N\log 2\left(\frac{\eta_2+\eta_3}{2}\right) + \frac{1}{\ln 2}\left(\sum_{j=1}^{N}\sum_{p=1}^{2N-j}\frac{1}{p} - \gamma N\right). \tag{42}$$

Comparing the RHS of (41) with that of (42), we note that $C_2^R \leq C_3^R$ if the following condition holds:

$$N\log 2\left(\frac{2\eta_1}{\eta_2+\eta_3}\right) \geq \frac{1}{\ln 2}\sum_{j=1}^{N}\sum_{p=N+1-j}^{2N-j}\frac{1}{p} \tag{43}$$

or equivalently

$$\frac{\eta_1}{\eta_2+\eta_3} \geq 2^{\frac{q}{N}-1}, \text{ where } q \triangleq \frac{1}{\ln 2}\sum_{j=1}^{N}\sum_{p=N+1-j}^{2N-j}\frac{1}{p}. \tag{44}$$

Combining (44) with the conditions in Lemma 5.1, we have that the ergodic capacity of the MIMO relay channel is C_2^R.

Recall that in Section IV-A, Case III reveals a somewhat surprising result that the upper bound meets the lower bound (in Case III, $\eta_2 = \eta_3$ and $N = 2$). Based on (44) and Lemma 5.1, it follows that if $\eta_1 \geq 3.2113\eta_2$ (which also implies $\eta_1 > \eta_3$), then $C_2^R < C_3^R$ and $C_2^R < C_1^R$. That is to say, the threshold value for this case is 3.2113, at which the upper bound and the lower bound "converge." Accordingly, in Fig. 5, where $\eta_1 = 10\eta_2$ and $\eta_2 = \eta_3$, the upper bound meets the lower bound and the ergodic capacity is achieved. To elaborate further on this, we present two different SNR parameters (cf. Case III in Section IV-A). As shown in Fig. 6, when $\eta_1 = 3\eta_2$, the upper bound is very close to the lower bound. Fig. 7 shows that the upper bound "meets" the lower bound perfectly when $\eta_1 = 3.5\eta_2$, indicating that the capacity can be characterized exactly.

C. Achievability of Ergodic Capacity: $N = 1$

If all the number of the antennas is one, the MIMO relay channel boils down to a scalar relay channel. In this case

$$C_2^R = \mathbb{E}_{\boldsymbol{h}}\log\left[1+\eta_2|h_2|^2+\eta_3|h_3|^2\right]$$
$$C_3^R = \mathbb{E}_{\boldsymbol{h}}\log\left[1+\eta_1|h_1|^2\right].$$

Since $|h_1|^2$ has χ^2 distribution with freedom of 2 (namely, exponential distribution), and $\mathbb{E}\left(|h_1|^2\right) = 1$, $X \triangleq \eta_1|h_1|^2$ has the probability density function

$$p_X(x) = \frac{1}{\eta_1}\exp\left(-\frac{x}{\eta_1}\right), \qquad x \geq 0.$$

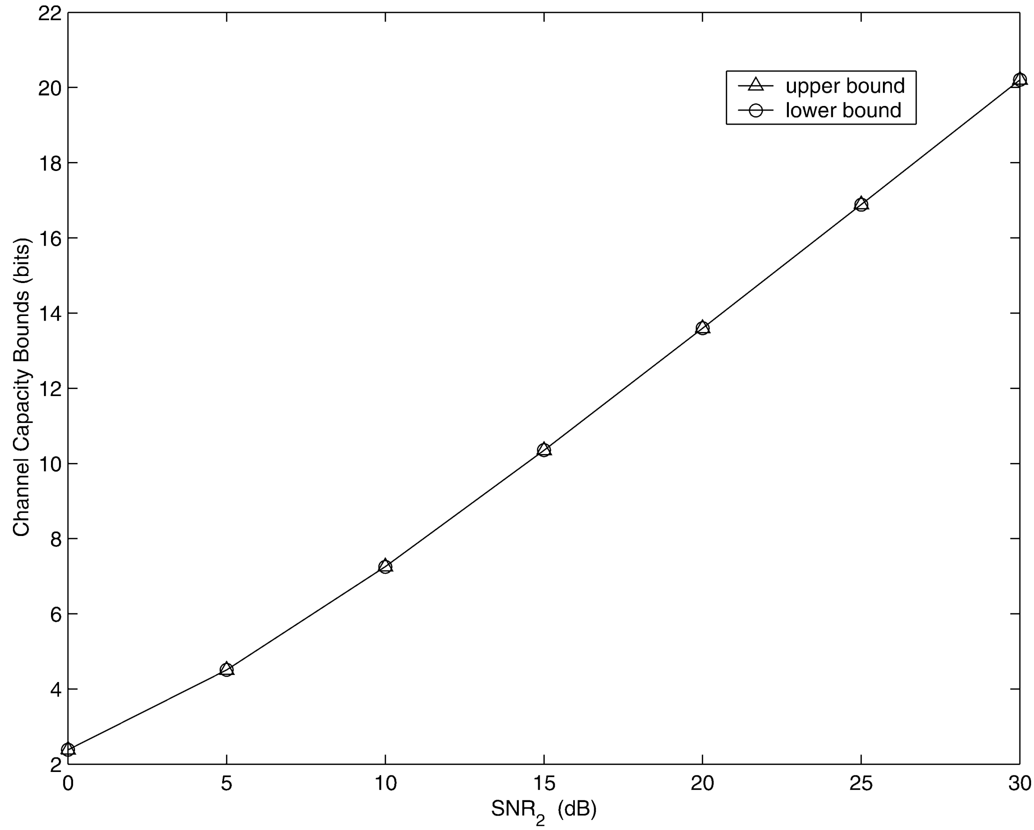

Fig. 7. Capacity bounds versus SNR_2 for the case of $\eta_2 = \eta_3$ and $\eta_1 = 3.5\eta_2$.

We can find C_3^R by

$$C_3^R = \frac{1}{\ln 2}\int_0^\infty \ln(1+x)\frac{1}{\eta_1}\exp\left(-\frac{x}{\eta_1}\right)dx. \tag{45}$$

Using the integral in [8, p. 568], we have that

$$C_3^R = -\frac{1}{\ln 2}e^{\frac{1}{\eta_1}}\mathrm{Ei}\left(-\frac{1}{\eta_1}\right) \tag{46}$$

where $\mathrm{Ei}(\cdot)$ is the exponential integral function in [8, p. 875].

Next, consider the random variable $Y \triangleq \eta_2|h_2|^2 + \eta_3|h_3|^2$. Since $\mathbb{E}\left(|h_2|^2\right) = \mathbb{E}\left(|h_3|^2\right) = 1$, we have that

$$\begin{aligned} p_Y(y) &= \int_0^y \frac{1}{\eta_2\eta_3}\exp\left(-\frac{y-t}{\eta_3}\right)\exp\left(-\frac{t}{\eta_2}\right)dt \\ &= \frac{1}{\eta_2-\eta_3}\left(e^{-\frac{y}{\eta_2}} - e^{-\frac{y}{\eta_3}}\right). \end{aligned} \tag{47}$$

Hence, C_2^R can be computed by

$$C_2^R = \frac{1}{\ln 2}\int_0^\infty \ln(1+y)\frac{1}{\eta_2-\eta_3}\left(e^{-\frac{y}{\eta_2}} - e^{-\frac{y}{\eta_3}}\right)dy \tag{48}$$

$$= \frac{1}{\ln 2}\frac{1}{\eta_2-\eta_3}\left[\eta_3 e^{\frac{1}{\eta_3}}\mathrm{Ei}\left(-\frac{1}{\eta_3}\right) - \eta_2 e^{\frac{1}{\eta_2}}\mathrm{Ei}\left(-\frac{1}{\eta_2}\right)\right]. \tag{49}$$

Combining (46) with (49), we conclude that if

$$\frac{1}{\eta_2-\eta_3}\left[\eta_3 e^{\frac{1}{\eta_3}}\mathrm{Ei}\left(-\frac{1}{\eta_3}\right) - \eta_2 e^{\frac{1}{\eta_2}}\mathrm{Ei}\left(-\frac{1}{\eta_2}\right)\right] \le e^{\frac{1}{\eta_1}}\mathrm{Ei}\left(-\frac{1}{\eta_1}\right) \tag{50}$$

and $\eta_1 \ge \eta_3$, then the ergodic capacity of the relay channel is given by

$$\boxed{C_2^R = \frac{1}{\ln 2}\frac{1}{\eta_2-\eta_3}\left[\eta_3 e^{\frac{1}{\eta_3}}\mathrm{Ei}\left(-\frac{1}{\eta_3}\right) - \eta_2 e^{\frac{1}{\eta_2}}\mathrm{Ei}\left(-\frac{1}{\eta_2}\right)\right].}$$

VI. An Application of MIMO Relay Channels in Cooperative Communications in *Ad Hoc* Networks

In what follows, we explore the utility of the MIMO relay channel for cooperative communications in *ad hoc* networks. We consider *ad hoc* networks using the IEEE 802.11 CSMA/CA standard. In such a context, the medium access control protocol uses the RTS (request-to-send)/CTS (clear-to-send) handshake to set up a communication link. More specifically, as shown in Fig. 8, source node S transmits an RTS packet to request the channel and destination node D replies with a CTS packet. If the RTS/CTS dialogue is successful, S and D begin their data communication, whereas all other nodes that hear either the RTS packet or the CTS packet are kept silent for a specified duration.

A key observation is that the silent node (node R in Fig. 8 within the shaded area) can be exploited to relay information. We use the capacity results on MIMO relay channel to characterize the performance gain therein over the direct transmission without relaying. The union of the transmission region of S and D is the so-called RTS/CTS reserved floor; R would have been kept silent while S communicates with D as RTS/CTS dialogue dictates; d is the distance between S and D. Note that if R lies in

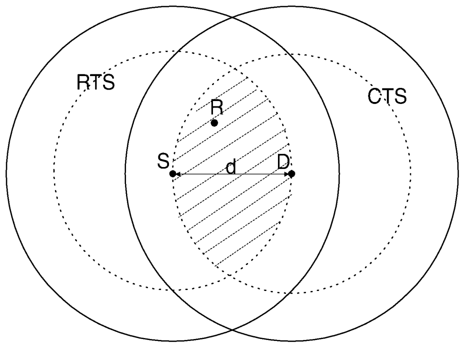

Fig. 8. A sketch of RTS/CTS dialogue.

the shaded area depicted in Fig. 8, it would have a shorter distance to both S and D than d. Thus, during the communication between S and D, R can function as a relay station to aid the data transmission. Thus motivated, we call this shaded area a "relay region," because any silent node within such a region can act as a relay node. Moreover, if R is equipped with multiple antennas, S, R, and D form a MIMO relay channel.

Consider the Rayleigh-fading channel. Define the relative gain of the capacity by using node R to relay data as

$$g \triangleq \left(C^R - C_d^R\right)/C_d^R$$

where C^R is the relay channel capacity, and C_d^R is the channel capacity corresponding to the direct link given in Theorem 4.2. Applying Theorems 4.1 and 4.2, we have that

$$\frac{C_{\text{lower}}^R - C_d^R}{C_d^R} \leq g \leq \frac{C_{\text{upper}}^R - C_d^R}{C_d^R}. \tag{51}$$

Observe that η_1 and η_3 depend on the coordinates of the relay node. Without loss of generality, let the coordinates of S, D, and R be $\left(-\frac{d}{2}, 0\right)$, $\left(\frac{d}{2}, 0\right)$, and (α, β). It follows that

$$\eta_1 = \eta_{10}\left[d/\sqrt{\left(\alpha + \frac{d}{2}\right)^2 + \beta^2}\right]^n \quad \text{and}$$
$$\eta_3 = \eta_{30}\left[d/\sqrt{\left(\alpha - \frac{d}{2}\right)^2 + \beta^2}\right]^n$$

where n is the path loss parameter in wireless links; and η_{10} and η_{30} are constants.

Since η_1 and η_3 change while the relay station moves, C_{lower} and C_{upper} are functions of (α, β) accordingly. Assume that R occurs with equal probability at any position within the shaded area (defined as $\mathcal{A}$) depicted in Fig. 8. Then it follows that

$$\begin{aligned}
\bar{C}_{\text{upper}}^R &= \frac{1}{S}\int\int_{(\alpha,\,\beta)\in\mathcal{A}_1} C_1^R(\alpha,\beta)\,d\alpha d\beta \\
&\quad + \frac{1}{S}\int\int_{(\alpha,\,\beta)\in\mathcal{A}_2} C_2^R(\alpha,\beta)\,d\alpha d\beta, \\
\bar{C}_{\text{lower}}^R &= \frac{1}{S}\int\int_{(\alpha,\,\beta)\in\mathcal{A}_3} C_d^R(\alpha,\beta)\,d\alpha d\beta \\
&\quad + \frac{1}{S}\int\int_{(\alpha,\,\beta)\in\mathcal{A}_4} \min\left(C_3^R(\alpha,\beta),\ C_2^R(\alpha,\beta)\right) \\
&\quad \times d\alpha d\beta
\end{aligned}$$

where S is the area of $\mathcal{A}$, given by $S = \left(\frac{2}{3}\pi - \frac{\sqrt{3}}{2}\right)d^2$; and $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$ are areas defined as

$$\begin{aligned}
\mathcal{A}_1 &\triangleq \left\{(\alpha,\beta)\,\middle|\,C_1^R \leq C_2^R\right\}\bigcap\mathcal{A} \\
\mathcal{A}_2 &\triangleq \left\{(\alpha,\beta)\,\middle|\,C_1^R > C_2^R\right\}\bigcap\mathcal{A} \\
\mathcal{A}_3 &\triangleq \left\{(\alpha,\beta)\,\middle|\,C_d^R \geq \min\left(C_3^R, C_2^R\right)\right\}\bigcap\mathcal{A} \\
\mathcal{A}_4 &\triangleq \left\{(\alpha,\beta)\,\middle|\,C_d^R < \min\left(C_3^R, C_2^R\right)\right\}\bigcap\mathcal{A}.
\end{aligned}$$

By using the lower bound and upper bound on the ergodic capacity of the relay channel, we have the corresponding lower and upper bounds on average relaying gain $\bar{g}$

$$\frac{\bar{C}_{\text{lower}}^R - C_d^R}{C_d^R} \leq \bar{g} \leq \frac{\bar{C}_{\text{upper}}^R - C_d^R}{C_d^R}. \tag{52}$$

VII. Conclusion

The relay channel is a basic model for multiuser communications in wireless networks. In this paper, we first study capacity bounds for the Gaussian MIMO relay channel with fixed channel gains. We derive an upper bound that involves convex optimization over two covariance matrices and *one* scalar parameter ρ. Loosely speaking, parameter ρ "captures" the cooperation between the source node and the relay node, and leads to solving the maximization problem using convex programming. We give an algorithm to computer the upper bound. We also present lower bounds on the MIMO relay channel capacity and provide algorithms to compute the bounds.

Next, we consider the Rayleigh fading case. We give an upper bound and a lower bound on the ergodic capacity. It is somewhat surprising that the upper bound can meet the lower bound under certain conditions (not necessarily degraded), indicating that the ergodic capacity can be characterized exactly. In particular, we identify sufficient conditions to achieve the ergodic capacity when all nodes have the same number of antennas; and our intuition for this finding is that the source node and the relay node can function as a "virtual" transmit antenna array when the relay node is located close to the source node, thus making it possible to achieve the capacity. Then we study the sufficient conditions under which the ergodic capacity can be characterized exactly. We examine two interesting cases, namely the high-SNR regime and the scalar relay channel case, and present the SNR conditions for achieving the capacity. The capacity results we obtain indicate independent coding schemes at the source node and the relay node; and our intuition is that the channel uncertainty (randomness) at transmitters results in such independence of the codebooks. We finally discuss a potential application of the MIMO relay channel in cooperative communications in *ad hoc* networks by using the capacity results.

We are currently pursuing to generalize the study to the partial transmitter CSI case.

Appendix A
Proof of Lemma 3.1

Since $\mathbf{\Sigma}_{22} \succ \mathbf{0}$, it follows from [11, 7.7.2] that

$$\boldsymbol{A}\boldsymbol{A}^\dagger = \mathbf{\Sigma}_{11}^{-\frac{1}{2}}\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{12}^\dagger\mathbf{\Sigma}_{11}^{-\frac{1}{2}}$$

is positive semidefinite. Therefore, we have that

$$I - AA^\dagger \preceq I_{M_1}$$

where "$\preceq$" is in terms of positive semidefinite ordering [11]. Given $X_2 = x_2$, the covariance matrix of X_1

$$\Sigma_{X_1|X_2=x_2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

is also positive semidefinite. It is easy to see that

$$I - AA^\dagger = \Sigma_{11}^{-\frac{1}{2}}\Sigma_{X_1|X_2=x_2}\Sigma_{11}^{-\frac{1}{2}} \succeq 0.$$

In summary, we have that

$$0 \preceq I - AA^\dagger \preceq I_{M_1}.$$

Thus, by the continuity of A in the vector space, there always exists $\rho \in [0,1]$ such that

$$I - AA^\dagger \preceq (1-\rho^2)I_{M_1}$$

or equivalently

$$AA^\dagger \succeq \rho^2 I_{M_1}. \quad (53)$$

The equality in (53) can be achieved if $M_1 \leq M_2$.

The converse of Lemma 3.1 holds by the continuity. That is to say, for a given ρ, there exists Σ_{12} such that $AA^\dagger = \rho^2 I_{M_1}$.

Appendix B
Another Lower Bound for the Fixed Channel Case

Following [4], we can find another lower bound on the capacity for the fixed channel case. By using block-Markov coding, the following rate is achieved for any given distribution $p(x_1, x_2)$:

$$R = \min\left(I(X_1;Y_1|X_2),\ I(X_1,X_2;Y)\right). \quad (54)$$

Recall that the optimal distribution is Gaussian [5], [4]; and the transmitted signal X_1 at the source node can be decomposed as

$$X_1 = \widetilde{X}_{10} + X_{11} \quad (55)$$

where, for brevity, we define $X_{11} \triangleq \Sigma_{12}\Sigma_{22}^{-1}X_2$. Based on the decomposition above and Lemma 3.1, the power constraints on $\widetilde{X}_{10}$ and X_{11} are given by

$$\mathbb{E}\left(\widetilde{X}_{10}^\dagger \widetilde{X}_{10}\right) \leq (1-\rho^2)M_1 \quad \text{and} \quad \mathbb{E}\left(X_{11}^\dagger X_{11}\right) \leq \rho^2 M_1. \quad (56)$$

We can choose the signal $\widetilde{X}_{10}$ to maximize the information rate for the source–relay link because $I(X_1;Y_1|X_2) = I(\widetilde{X}_{10};Y_1)$. Then the corresponding solution for $\widetilde{X}_{10}$ can be found by using the water-filling technique [23]. More specifically, let the singular value decomposition (SVD) of H_1 be $U_1\Lambda_1V_1^\dagger$. Accordingly, the water-filling solution for $\widetilde{X}_{10}$ is given by

$$\widetilde{X}_{10} = V_1T_{10} \quad (57)$$

where T_{10} is a circularly symmetric complex Gaussian vector satisfying

$$\mathbb{E}\left(T_{10}T_{10}^\dagger\right) = D_{10} \quad (58)$$

with D_{10} being a diagonal matrix (see [23]).

Now consider the multiple-access link from the source node and the relay node to the destination node, i.e., $(X_1, X_2) \to Y$. We can view this as a MIMO channel with $M_1 + M_2$ transmit antennas and N receive antennas. Define $m = \min(M_1 + M_2,\ N)$. Since $I(X_1, X_2; Y) = I(X_{11}, X_2; Y)$, we can choose the joint distribution $p(X_{11}, X_2)$ to maximize the sum information rate for the multiple-access link. To this end, the received signal at the destination node can be written as

$$Y = \left[\sqrt{\eta_2}H_2 \quad \sqrt{\eta_3}H_3\right]\begin{bmatrix} X_{11} \\ X_2 \end{bmatrix} + \sqrt{\eta_2}H_2V_1T_{10} + Z. \quad (59)$$

Let L^{-1} be a whitening matrix for $\sqrt{\eta_2}H_2V_1T_{10} + Z_1$, and define

$$H_w \triangleq L^{-1}\left[\sqrt{\eta_2}H_2 \quad \sqrt{\eta_3}H_3\right].$$

Let $U_w\Lambda_wV_w^\dagger$ be the SVD of H_w. Rewrite the transmitted signal as

$$\begin{bmatrix} X_{11} \\ X_2 \end{bmatrix} = V_wT_w \quad (60)$$

where T_w is a circularly symmetric complex Gaussian vector satisfying

$$\mathbb{E}\left(T_wT_w^\dagger\right) = D_w \quad (61)$$

with D_w being a diagonal matrix (see [23]).

Then the sum information rate for the multiple-access link of the MIMO relay channel is given by

$$I(X_1, X_2; Y) = \log\left[\det\left(I + \Lambda_wD_w\Lambda_w^\dagger\right)\right] + \log\left[\det\left(LL^\dagger\right)\right]. \quad (62)$$

This is a water-filling problem with respect to D_w except that each terminal has its own power constraint. Let V_{w_1} denote the M_1 first rows of V_w, and V_{w_2} the remaining rows. Then the power constraints are as follows:

$$\operatorname{tr}\left(V_{w1}D_wV_{w1}^\dagger\right) = \sum_{k=1}^{m} c_{1k}[D_w]_{kk} \leq \rho^2 M_1 \quad (63)$$

$$\operatorname{tr}\left(V_{w2}D_wV_{w2}^\dagger\right) = \sum_{k=1}^{m} c_{2k}[D_w]_{kk} \leq M_2 \quad (64)$$

where c_{1k} is the nonzero eigenvalue of $V_{w_1}V_{w_1}^\dagger$, and c_{2k} is the nonzero eigenvalue of $V_{w_2}V_{w_2}^\dagger$; and $[D_w]_{kk}$ denotes the kth element along the diagonal of D_w.

The problem can be solved by using the Lagrange dual function

$$\mathcal{L}(\lambda_1,\ \lambda_2) = \sum_{k=1}^{m} \log(1 + \left[\Lambda_w^\dagger\Lambda_w\right]_{kk}[D_w]_{kk}) + \lambda_1\left(\sum_{k=1}^{m} c_{1k}[D_w]_{kk} - \rho^2M_1\right) + \lambda_2\left(\sum_{k=1}^{m} c_{2k}[D_w]_{kk} - M_2\right) \quad (65)$$

where λ_1 and λ_2 are Lagrange multipliers.

Alternatively, we can find a suboptimal solution by first finding the standard water-filling solution subject to a total power constraint, following by scaling this to satisfy the individual power constraints. That is to say, rewrite

$$\begin{bmatrix} \boldsymbol{X}_{11} \\ \boldsymbol{X}_2 \end{bmatrix} = \begin{bmatrix} \alpha 1 \boldsymbol{V}_{w1} \\ \alpha 2 \boldsymbol{V}_{w2} \end{bmatrix} \boldsymbol{T}_w \tag{66}$$

where α_1 and α_2 satisfy

$$\alpha 1 \mathrm{tr} \begin{bmatrix} \boldsymbol{V}_{w1} & \boldsymbol{D}_w & \boldsymbol{V}_{w1}^{\dagger} \end{bmatrix} = \rho^2 M_1 \tag{67}$$

$$\alpha 2 \mathrm{tr} \begin{bmatrix} \boldsymbol{V}_{w2} & \boldsymbol{D}_w & \boldsymbol{V}_{w2}^{\dagger} \end{bmatrix} = M_2. \tag{68}$$

Then, we can maximize the achievable rate with respect to parameter ρ after plugging D_{01} and D_w into (54).

ACKNOWLEDGMENT

The authors wish to thank Dr. Gerhard Kramer for [13] and helpful comments on the proof of Theorem 4.1. They would also like to thank the anonymous reviewers for their helpful comments that greatly improved the presentation of the paper.

REFERENCES

[1] (2004) Convex Opitmization. Stephen Boyd and Lieven Vandenberghe. [Online]. Available: http://www.stanford.edu/ boyd/cvxbook.html

[2] G. Caire and S. Shamai (Shitz), "On the achievable throughput of a multi-antenna Gaussian broadcast channel," *IEEE Trans. Inf. Theory*, vol. 49, no. 7, pp. 1691–1706, Jul. 2003.

[3] R. S. Cheng and S. Verdú, "Gaussian multiaccess channels with ISI: Capacity region and multiuser water-filling," *IEEE Trans. Inf. Theory*, vol. 39, no. 3, pp. 773–785, May 1993.

[4] T. M. Cover and A. A. El Gamal, "Capacity theorems for the relay channel," *IEEE Trans. Inf. Theory*, vol. IT-25, no. 6, pp. 572–584, Sep. 1979.

[5] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York: Wiley, 1991.

[6] A. A. El Gammal, *Lecture Notes 7: Relay Channel*. Stanford, CA: EEE478 Stanford Univ. , 2002.

[7] M. Gastpar, G. Kramer, and P. Gupta, "The multiple-relay channel: Coding and antenna-clustering capacity," in *Proc. IEEE Int. Symp. Information Theory (ISIT 2002)*, Lausanne, Switzerland, Jun./Jul. 2002, p. 136.

[8] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*. San Diego, CA: Academic, 2000.

[9] A. K. Gupta and D. K. Nagar, *Matrix Variate Distributions*. London, U.K.: Champman & Hall, 2000.

[10] A. Høst-Madsen and J. Zhang, "Capacity bounds and power allocation for wireless relay channel," *IEEE Trans. Inf. Theory*, submitted for publication.

[11] R. A. Horn and G. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1999.

[12] G. Kramer, M. Gastpar, and P. Gupta, "Capacity theorems for wireless relay channels," in *Proc. Allerton Conf. Communication, Control, and Computeing (Allerton 2003)*, Monticello, IL, Oct. 2003, pp. 1074–1083.

[13] G. Kramer, P. Gupta, and M. Gastpar, "Information-theoretic multi-hopping for relay networks," in *Proc. 2004 Int. Zurich Seminar*, Zurich, Switzerland, Feb. 2004.

[14] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior," *IEEE Trans. Inf. Theory*, vol. 50, no. 12, pp. 3062–3080, Dec. 2004.

[15] ——, "Distributed space-time-coded protocols for exploiting cooperative diversity in wireless networks," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2415–2425, Oct. 2003.

[16] T. L. Marzetta and B. M. Hochwald, "Capacity of a mobile mulit-antenna communication link in Rayleigh flat fading," *IEEE Trans. Inf. Theory*, vol. 45, no. 1, pp. 139–157, Jan. 1999.

[17] R. U. Nabar, H. Bölcskei, and F. W. Kneubühler, "Fading relay channels: Performance limits and space-time signal design," *IEEE J. Select. Areas Commun.*, to be published.

[18] R. U. Nabar, Ö. r Oyman, H. Bölcskei, and A. J. Paulraj, "Capacity scaling laws in MIMO wireless networks," in *Proc. Allerton Conf. Communication, Control, and Computeing (Allerton 2003)*, Monticello, IL, Oct. 2003, pp. 378–389.

[19] Ö. r Oyman, R. U. Nabar, H. Bölcskei, and A. J. Paulraj, "Characterizing the statistical properties of mutual information in MIMO channels," *IEEE Trans. Signal Porcessing*, vol. 51, no. 11, pp. 2784–2795, Nov. 2003.

[20] W. Rhee and J. M. Cioffi, "Ergodic capacity of multi-antenna Gaussian multiple access channels," in *Proc. 35th Asilomar Conf. Signals, Systems and Computers*, Pacific Grove, CA, Nov. 2001.

[21] A. Sendonaris, E. Erkip, and B. Aazhang, "User cooperation diversity-part I: System description," *IEEE Trans. Commun.*, vol. 51, no. 11, pp. 1927–1938, Nov. 2003.

[22] ——, "User cooperation diversity-part II: Implementation aspects and performance analysis," *IEEE Trans. Commun.*, vol. 51, no. 11, pp. 1939–1948, Nov. 2003.

[23] S. K. Jayaweera and H. V. Poor, "Capacity of mutliple-antenna systems with both receiver and transmitter channel state information," *IEEE Trans. Inform. Theory*, vol. 49, no. 10, pp. 2697–2708, Oct. 2003.

[24] İ. E. Telatar, "Capacity of mulit-antenna Gaussian channels," *Europ. Trans. Telecommun.*, vol. 10, pp. 585–595, Nov. 1999.

[25] S. Vishwanath, N. Jindal, and A. Goldsmith, "Duality, achievable rates and sum-rate capacity of Gaussian MIMO broadcast channels," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2658–2668, Oct. 2003.

[26] P. Viswanath and D. N. C. Tse, "Sum capacity of the vector Gaussian broadcast channel and uplink–downlink duality," *IEEE Trans. Inf. Theory*, vol. 49, no. 8, pp. 1912–1921, Aug. 2003.

[27] B. Wang and J. Zhang, "MIMO relay channel and its application for cooperative communication in ad hoc networks," in *Proc. Allerton Conf. Communication, Control, and Computeing (Allerton 2003)*, Monticello, IL, Oct. 2003, pp. 1556–1565.

[28] B. Wang, J. Zhang, and A. Høst-Madsen, "On ergodic capacity of MIMO relay channel," in *Proc. Conf. Information Sciences and Systems (CISS 2004)*, Princeton, NJ, Mar. 2004, pp. 603–608.

[29] H. Weingarten, Y. Steinberg, and S. Shamai (Shitz), "The capacity region of the Gaussian MIMO broadcast channel," in *Proc IEEE Int. Symp. Information Theory (ISIT2004)*, Chicago, IL, Jun./Jul. 2004, p. 174.

[30] A. D. Wyner, "Shannon-theorectic approach to a Gaussian cellular multiple-access channel," *IEEE Trans. Inf. Theory*, vol. 41, no. 6, pp. 1713–1727, Nov. 1994.

[31] L.-L. Xie and P. R. Kumar, "A network information theory for wireless communication: Scaling laws and optimal operation," *IEEE Trans. Inf. Theory*, vol. 50, no. 5, pp. 748–767, May 2004.

[32] W. Yu and J. M. Cioffi, "Sum capacity of a Gaussian vector broadcast channel," *IEEE Trans. Inf. Theory*, submitted for publication.

[33] W. Yu, W. Rhee, S. Boyd, and J. M. Cioffi, "Iterative water-filling for Gaussian vector multiple access channels," *IEEE Trans. Inf. Theory*, vol. 50, no. 1, pp. 145–152, Jan. 2004.

[34] L Zheng, "Diversity-multiplexing tradeoff: A comprehensive view of multiple antenna systems," Ph.D. dissertation, Univ. California, Berkeley, 2002.