that the estimate is universally consistent, that is, $\mathbb{E}\|f_n - f\| \to 0$ for any density. Also, Devroye [3] shows that for any $\epsilon > 0$

$$\mathbb{P}\{\|f_n - f\| - \mathbb{E}\|f_n - f\| \geq \epsilon\} \leq e^{-n\epsilon^2/2}.$$

Using these properties, it is easy to see that the testing method based on the kernel density estimate is consistent in the sense that the probability of error converges to zero exponentially for all $f \in \bigcup_{j=1}^{k} \overline{H}_j$. In order to show this, suppose that $f \in \overline{H}_1$, and put

$$\epsilon = \min_{j>1} \left\|f - f^{(j)}\right\| - \left\|f - f^{(1)}\right\|.$$

Then

$$\begin{aligned}
\mathbb{P}\{\text{error}\} &\leq \mathbb{P}\left\{\exists\, j > 1\colon \left\|f_n - f^{(1)}\right\| \geq \left\|f_n - f^{(j)}\right\|\right\} \\
&\leq (k-1)\max_{j>1} \mathbb{P}\left\{\left\|f_n - f^{(1)}\right\| \geq \left\|f_n - f^{(j)}\right\|\right\} \\
&\leq (k-1)\max_{j>1} \mathbb{P}\left\{\|f_n - f\| + \left\|f - f^{(1)}\right\| \right. \\
&\qquad\qquad \left. \geq \left\|f - f^{(j)}\right\| - \|f_n - f\|\right\} \\
&\leq (k-1)\mathbb{P}\{2\|f_n - f\| \geq \epsilon\} \\
&= (k-1)\mathbb{P}\{\|f_n - f\| - \mathbb{E}\|f_n - f\| \\
&\qquad\qquad \geq \epsilon/2 - \mathbb{E}\|f_n - f\|\} \\
&\leq (k-1)e^{-n/2([\epsilon/2 - \mathbb{E}\|f_n - f\|]^+)^2}
\end{aligned}$$

where the last inequality follows from the previously mentioned inequality of Devroye [3]. The consistency of f_n assures that for a sufficiently large n, $\mathbb{E}\|f_n - f\| < \epsilon/4$ and for such n, $\mathbb{P}\{\text{error}\} \leq (k-1)e^{-n\epsilon^2/32}$. However, since $\mathbb{E}\|f_n - f\|$ may tend to zero at an arbitrarily slow rate (see [2]), the error exponent is not uniform: it depends on f. It is known (see [1], [12]) that for the hypotheses $\overline{H}_j$ it is impossible to construct a test with a uniform error exponent.

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Linear MMSE Multiuser Receivers: MAI Conditional Weak Convergence and Network Capacity

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***Abstract*—We explore the performance of minimum mean-square error (MMSE) multiuser receivers in wireless systems where the signatures are modeled as random and take values in complex space. First we study the conditional distribution of the output multiple-access interference (MAI) of the MMSE receiver. By appealing to the notion of *conditional weak convergence*, we find that the conditional distribution of the output MAI, given the received signatures and received powers, converges in probability to a proper complex Gaussian distribution that does not depend on the signatures. This result indicates that, in a large system, the output interference of the MMSE receiver is approximately Gaussian with high probability, and that systems with MMSE receivers are robust to the randomness of the signatures. Building on the Gaussianity of the output interference, we then take the quality of service (QoS) requirements as meeting the signal-to-interference ratio (SIR) constraints and identify the network capacity of single-class systems with random spreading. The network capacity is expressed uniquely in terms of the SIR requirements and received power distributions. Compared to the network capacity corresponding to the optimal signature allocation, we conclude that at the cost of transmission power, the gap between the network capacity corresponding to optimal signatures and that corresponding to random signatures can be made arbitrarily small. Therefore, from the viewpoint of network capacity, systems with MMSE receivers are robust to the randomness of signatures.**

***Index Terms*—Central limit theorem, conditional weak convergence, martingale difference array, minimum mean-square error (MMSE) receiver, proper complex random variable, random signature.**

I. Introduction

Consider a K-user communication system equipped with linear minimum mean-square error (MMSE) multiuser receivers.[1] We focus primarily on the following discrete-time synchronous baseband model

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[1] As in [3], we assume coherent demodulation in this paper.

[15]–[17]. In a symbol interval, the received signal at the front end of the receiver is

$$Y^{(N)} = \sum_{i=1}^{K} \sqrt{P_i}\, b_i s_i + V \tag{1}$$

where the b_i's are the transmitted (complex) information symbols, the P_i's are the received powers, the s_i's ($s_i = \frac{1}{\sqrt{N}}[s_{i1}, \ldots, s_{iN}]^t$) are the signatures, and V comes from the sampling of the *proper complex* white Gaussian noise with power spectral density η.[2] (We assume throughout that $\eta > 0$.)

The signatures provide a mechanism for separating users at the receiver and the corresponding model is of considerable interest (see, e.g., [6], [15], [16]). In this correspondence, we assume that the signatures take values in complex space. We focus on systems with random spreading, as in [15], [17], [19], [20]. The random signature model is applicable to many practical systems. For example, this model is applicable to code-division multiple-access (CDMA) systems with very long pseudorandom spreading sequences, and also applicable to CDMA systems with random short signatures (i.e., the period of the signature equals the information symbol period, and repetition of the same random signatures is adopted). In CDMA systems, the number of chips per signature N is sometimes called the *processing gain*. This model is also applicable to multiple-antenna systems where the vector s_i represents the fading levels of user i at each of the N antennas. In this case, we call s_i the *spatial fading signature* of user i. The s_i's in a multiple-antenna system are often modeled to be random and take values in N-dimensional complex space. The independence of the spatial fading signatures can be achieved by physically separating the antennas by a few carrier wavelengths [7]. We note that in both CDMA and multiple-antenna systems, the random signature model can also account for the random phase rotation due to fading. Regardless of the specific applications, we call the length of the signatures the *degree of freedom*. More specifically, the degree of freedom is the processing gain in a CDMA system or the number of antennas in a multiple-antenna system [15].

We consider a more realistic scenario where users transmit data through a fading channel, and each user is capable of decentralized power control. Accordingly, we assume that the received powers are random (due to imperfect power control), and are independent across the users. We denote the received power of user i as P_i and its mean μ_i. Our results are asymptotic in nature, with both K and N going to infinity. As we scale up the system (as $N \to \infty$), the ratio of K to N is denoted by $\alpha \triangleq \frac{K}{N}$ and taken to be fixed, as is standard (see, e.g., [15], [17], [19]).

In this correspondence, we first study the distribution of the multiple-access interference (MAI) at the output of the MMSE receiver. The output MAI distribution is a physical-layer performance metric and plays a crucial role in determining bit error probability. We focus on systems where repetition of the same random signatures is adopted and the received powers change relatively slowly compared to the symbol rate. In these systems, it is of more interest to study the conditional distributions of the output MAI given the signatures and received powers. Our study makes use of the notion of *conditional weak convergence* [14]. In particular, our analysis involves two modes of conditional weak convergence—*convergence almost surely* of conditional distributions and *convergence in probability* of conditional distributions. Roughly speaking, our main result on the output MAI distributions can be summarized as follows.

[2]Proper complex random processes are also known as *circularly symmetric* random processes [4]. We elaborate further on proper complex random processes in Section III.

- Assuming that the empirical distribution function[3] of the mean received powers $\{\mu_1, \ldots, \mu_K\}$ converges weakly as $N \to \infty$, the conditional distribution of the output MAI of the MMSE receiver, given the received signatures and received powers, converges in probability to a proper complex Gaussian distribution that does not depend on the signatures.

The above result can be viewed as a generalization of [20, Theorems 3.1 and 3.2], which established the Gaussianity of the output MAI under the assumption that the signatures are binary spreading sequences. However, the technical nature of the above result is significantly different from that of [20, Theorem 3.2]. Indeed, the relaxation of the signatures to be complex gives rise to possibly much more variation in the MAI, requiring the use of the new notion of conditional weak convergence. We note that assuming signatures are deterministic, Poor and Verdú [12] have established the Gaussianity of the output interference of the MMSE receiver under several asymptotic conditions (the output MAI vanishes in these scenarios).

Next we take a network perspective and identify the network capacity of single-class systems. Loosely speaking, a set of users is admissible if their simultaneous transmission does not result in violation of any of their quality of service (QoS) requirements; the *network capacity* is the maximum number of admissible users. Building on the Gaussianity of the output MAI, we take the QoS requirements as meeting the signal-to-interference ratio (SIR) constraints. Our result shows that the network capacity can be expressed uniquely in terms of the SIR requirements and received power distributions. The network capacity of imperfect power-controlled systems with linear receivers has been studied in [19]. In particular, in [19] the authors characterized the network capacity of systems with MMSE receivers for the deterministic signature case and the corresponding characterization for the random signature case was left open. We resolve this problem in this correspondence. Combining these results, we observe that at the cost of transmission power, we can drive the gap between the network capacity corresponding to optimal signatures and that corresponding to random signatures arbitrarily small.

The organization of the remainder of this correspondence is as follows. The next section contains our model description. In Section III, we provide some necessary mathematical background. We present our main results in Section IV, and the proofs of our main results are relegated to Section V. We draw our conclusions in Section VI.

II. Model Description

Consider a canonical discrete-time symbol-synchronous multiple-access model. In a symbol interval, the received signal before filtering is

$$Y^{(N)} = \sum_{i=1}^{K} \sqrt{P_i}\, b_i s_i + V.$$

Without loss of generality, we consider user 1. The MMSE receiver exploits the MAI structure provided by the signatures and received powers of the interferers. Because the received powers may vary from symbol to symbol, we assume that the MMSE receiver has knowledge of the mean received powers instead of the instantaneous received powers of the interferers [20]. Also, assume for now that the MMSE receiver has knowledge of P_1, the instantaneous received power of user 1.

[3]See [2, p. 279] and [1, p. 268] for the definitions of empirical distribution functions.

(It turns out that there is really no need for knowledge of P_1. We will return to elaborate on this issue in Section IV.) Define

$$S_1 \triangleq [s_2, \ldots, s_K], \qquad S \triangleq [s_1, s_2, \ldots, s_K]$$
$$D_1 \triangleq \mathrm{diag}(P_2, \ldots, P_K), \quad E_1 \triangleq \mathrm{diag}(\mu_2, \ldots, \mu_K)$$
$$M_I \triangleq S_1 E_1 S_1^H + \eta I, \qquad M_I' \triangleq S_1 D_1 S_1^H + \eta I.$$

The MMSE receiver generates an output of the form $c^H Y^{(N)}$, where c is chosen to minimize the mean-square error

$$J = \mathbb{E}\left[\left|c^H Y^{(N)} - b_1\right|^2 \middle| P_1, S\right].$$

It is straightforward to see that the MMSE receiver is $\sqrt{P_1}\, M^{-1} s_1$, where $M \triangleq P_1 s_1 s_1^H + M_I$. After some algebra, the output at the MMSE multiuser receiver can be expressed as follows (cf. [5], [15], [19]):

$$y_1^{(N)} = \frac{P_1 s_1^H M_I^{-1} s_1}{1 + P_1 s_1^H M_I^{-1} s_1}\, b_1 + \mathcal{I}_1^{(N)} + \mathcal{I}_2^{(N)} \tag{2}$$

where

$$\mathcal{I}_1^{(N)} = \sum_{i=2}^{K} \frac{\sqrt{P_1}}{1 + P_1 s_1^H M_I^{-1} s_1}\, s_1^H M_I^{-1} \sqrt{P_i}\, b_i s_i$$
$$\mathcal{I}_2^{(N)} = \frac{\sqrt{P_1}}{1 + P_1 s_1^H M_I^{-1} s_1}\, s_1^H M_I^{-1} V.$$

We note that $\mathcal{I}_1^{(N)}$ denotes the output MAI and $\mathcal{I}_2^{(N)}$ denotes the effect of background noise.

As is standard (see, e.g., [5]), the SIR is defined to be the ratio of the desired signal power to the sum of the noise and MAI powers at the output of the MMSE receiver in a symbol interval. Thus, we have that the SIR of user 1 is

$$\mathrm{SIR}_1^{(N)} = \frac{P_1 \left(s_1^H M_I^{-1} s_1\right)^2}{s_1^H M_I^{-1} M_I' M_I^{-1} s_1}. \tag{3}$$

As will be shown in Theorem 4.1, in a large system the output interference of the MMSE receiver can be well approximated as Gaussian. Therefore, it is reasonable to take the QoS requirement as meeting the SIR constraints (see, e.g., [15], [17], [19]). Because of the randomness of the received powers and signatures, the SIR is random as well. Thus, motivated, we adopt a probabilistic model for the users' QoS requirements as follows (cf. [10])

$$P\left\{\mathrm{SIR}_k^{(N)} \ge \gamma_k\right\} > a_k$$

where $\mathrm{SIR}_k^{(N)}$ is the SIR of user k when the degree of freedom is N, γ_k the *target SIR* of user k, and $a_k \in [0, 1)$, $k = 1, \ldots, K$. That is, the probability that the SIR of user k is no less than γ_k must be greater than a_k.

In this correspondence, we are interested in identifying the maximum number of users admissible in a single-class system. A single-class system is one where the received powers of all the users are independent and identically distributed (i.i.d.) (see [15], [19]). We note that this model does arise in practical situations—for example, in a power-controlled cellular system with only voice users.

Because the received powers are i.i.d. and the signatures are i.i.d., the SIR is identically distributed for all the users. Then, $P\{\mathrm{SIR}_k^{(N)} \ge \gamma\}$ does not depend on k. Following [19], we define the *network capacity* of a system with linear MMSE receivers (when the degree of freedom is N) as follows:

$$\alpha_N(a) \triangleq \sup\left\{\alpha \,\middle|\, P\left\{\mathrm{SIR}_1^{(N)} \ge \gamma\right\} > a\right\}.$$

The *asymptotic network capacity* $\alpha_\infty(a)$ is defined as the limit of $\{\alpha_N(a)\}$

$$\alpha_\infty(a) \triangleq \lim_{N \to \infty} \alpha_N(a).$$

III. Mathematical Preliminaries

A. Two Modes of Conditional Weak Convergence

For convenience, we let $X^{(N)}$ denote the collection of the signatures and received powers when the degree of freedom is N, and $\boldsymbol{X}$ the sequence $\{X^{(N)}\}$. We are primarily interested in the asymptotic distributions of the $\mathcal{I}_1^{(N)}$'s given $\boldsymbol{X}$. Because the conditional distributions of the $\mathcal{I}_1^{(N)}$'s given $\boldsymbol{X}$ are random probability measures, the convergence of conditional distributions involves *conditional weak convergence*. In what follows, we refer to [14] and briefly restate a few definitions of modes of conditional weak convergence.

It is clear that the output MAI, $\mathcal{I}_1^{(N)}$, takes values in the complex space $\mathbb{C}$. Let M_1 be the space of all measures on $(\mathbb{C}, \mathcal{B}_\mathbb{C})$, where $\mathcal{B}_\mathbb{C}$ is the Borel σ-algebra on $\mathbb{C}$, and $\mathcal{M}_1$ is the σ-algebra generated by the weak topology on M_1. Let $C(\mathbb{C})$ denote the space of bounded continuous functions on $\mathbb{C}$. Let $P_N^{\boldsymbol{X}}$ denote the conditional distribution of $\mathcal{I}_1^{(N)}$ given $\boldsymbol{X}$. We define *almost sure convergence* of conditional distributions as follows.

Definition 3.1: The conditional distribution of $\mathcal{I}_1^{(N)}$ given $\boldsymbol{X}$ converges almost surely to a random measure P_ν in $(M_1, \mathcal{M}_1)$, written as $P_N^{\boldsymbol{X}} \overset{a.s.}{\Longrightarrow} P_\nu$, if for all $f \in C(\mathbb{C})$, $\int f\, dP_N^{\boldsymbol{X}} \to \int f\, dP_\nu$ with probability one.

As pointed out in [14], the most convenient formulation of *convergence in probability* of conditional distributions is in terms of almost sure convergence of subsequences of conditional distributions. We state its definition in the following.

Definition 3.2: The conditional distribution of $\mathcal{I}_1^{(N)}$ given $\boldsymbol{X}$ converges in probability to a random measure P_ν in $(M_1, \mathcal{M}_1)$, written as $P_N^{\boldsymbol{X}} \overset{P}{\Longrightarrow} P_\nu$, if every subsequence $\{N'\}$ contains a further subsequence $\{N''\}$ for which $P_{N''}^{\boldsymbol{X}} \overset{a.s.}{\Longrightarrow} P_\nu$.

Note that in the above, for brevity we have used the notation $\{N'\}$ and $\{N''\}$ to represent the subsequences $\{P_{N'}^{\boldsymbol{X}}\}$ and $\{P_{N''}^{\boldsymbol{X}}\}$, respectively.

Because $\mathbb{C}$ is separable and hence M_1 is metrizable, the above definition coincides with the standard definition of convergence in probability in a metric space. Indeed, this conclusion can be further illustrated by the following important result on convergence in probability of random variables $\{X_n\}$ [1, Theorem 20.5].

Lemma 3.1: A necessary and sufficient condition for $X_n \overset{P}{\longrightarrow} X$ is that each subsequence $\{X_{n'}\}$ contains a further subsequence $\{X_{n''}\}$ such that $X_{n''} \to X$ with probability one.

We shall use the above result repeatedly in the proof of our main results.

B. Proper Complex Random Variables

We begin with a few definitions.

Definition 3.3: A complex random variable is defined as a random variable of the form

$$t = t_c + j t_s, \qquad j = \sqrt{-1},$$

where t_c and t_s are real random variables defined on the same probability space [9, p. 14].

As shown in [11], the "covariance" of two complex random variables $t = t_c + j t_s$ and $r = r_c + j r_s$ (where the "covariance" refers to the four covariances arising between the real and imaginary parts of t and r), when defined consistently with the corresponding notion for real random variables, is determined by the usual (complex) covariance together with a quantity called the *pseudocovariance*. The covariance $\Sigma_{t,r}$ and pseudocovariance $\tilde{\Sigma}_{t,r}$ are defined as follows:

$$\Sigma_{t,r} \triangleq \mathbb{E}\left[(t-\mathbb{E}[t])\overline{(r-\mathbb{E}[r])}\right] \tag{4}$$

$$\tilde{\Sigma}_{t,r} \triangleq \mathbb{E}[(t-\mathbb{E}[t])(r-\mathbb{E}[r])]. \tag{5}$$

Definition 3.4: A complex random variable $t = t_c + jt_s$ is *proper* if its pseudocovariance $\tilde{\Sigma}_t = \mathbb{E}[(t-\mathbb{E}[t])^2]$ vanishes [11].

Proper complex random variables are also known as *circularly symmetric* complex random variables. It should be noted that t is proper if and only if t_c and t_s have the same variance and are uncorrelated. As shown in [11], the complex multivariate Gaussian density assumes a natural form only for proper complex Gaussian random variables. We call the distribution of a proper complex Gaussian random variable a *proper complex Gaussian distribution*.

C. Regularity Conditions

We assume that the signatures are chosen randomly and independently. The model for random signatures is as follows:

$$s_i = \frac{1}{\sqrt{N}}[s_{i1}, \ldots, s_{iN}]^T$$

where the s_{in}'s are i.i.d. proper complex random variables with zero mean and covariance 1, that is, $\mathbb{E}[|s_{in}|^2] = 1$. For technical reasons, we further assume that $\mathbb{E}[|s_{in}|^4] < \infty$.

We assume that the P_i's and the b_i's are independent. The assumptions we impose on the received powers and information symbols are listed as follows.

3.P1) The empirical distribution function of $\{\mu_1, \ldots, \mu_K\}$ converges weakly to a distribution function H_μ as $N \to \infty$.

3.P2) The P_i's are uniformly bounded above and the μ_i's are bounded below by a positive number.

3.P3) The b_i's are independent proper complex random variables with $\mathbb{E}[b_i] = 0$ and $\mathbb{E}[|b_i|^2] = 1$, and the fourth moments of the b_is are bounded.

We note that Condition 3.P1) is a standard assumption imposed in the literature on large system analysis (see, e.g., [15]). Indeed, this condition holds in many practical scenarios. For instance, when the received powers of all users follow the same distribution, the corresponding mean μ is the same for all users, and the limiting empirical distribution is the one with $P(X=\mu) = 1$. Condition 3.P2) on the received powers is sensible since the received power is bounded in any practical system. Condition 3.P3) on the signal constellation covers many modulation methods of interest, for example, quaternary phase-shift keying (QPSK) modulation schemes.

IV. Summary of Main Results

In this section, we summarize the main contributions of this correspondence. The proofs of our results are relegated to Section V.

Our first main result is on the asymptotic distributions of the output MAI conditioned on the signatures and received powers. This result is a generalization of [20, Theorems 3.1 and 3.2], which established the Gaussianity of the output MAI under the assumption that the signatures are binary spreading sequences. We note that although the proof of Theorem 4.1, part a) has some similarity in flavor to that of [20, Theorem 3.1], the technical nature of our proof for Theorem 4.1, part b), the main part of Theorem 4.1, is significantly different from that of [20, Theorem 3.2]. Indeed, the relaxation of the signatures to be complex gives rise to possibly much more variation in the MAI. We use conceptually the more subtle notion of *conditional weak convergence* to resolve this problem.

Theorem 4.1: Suppose Conditions 3.P1)–3.P3) hold. Then we have the following (as $N \to \infty$).

a) The output MAI of the MMSE receiver, $\mathcal{I}_1^{(N)}$, has a limiting proper complex Gaussian distribution.

b) The conditional distribution of the output MAI of the MMSE receiver, given the received signatures and received powers, converges in probability to the same proper complex Gaussian distribution as in part a).

In Theorem 4.1, part a) shows that the unconditional distribution of $\mathcal{I}_1^{(N)}$ converges (in the weak sense) to a proper complex Gaussian distribution; part b) establishes that the conditional distribution of $\mathcal{I}_1^{(N)}$, given $\boldsymbol{X}$, converges (in the weak sense) to the same proper complex Gaussian distribution in probability (see also Definition 3.2). We note that the limiting proper complex Gaussian distribution does not depend on the signatures and depends "weakly" on the received powers in the sense that its variance depends on only the empirical distribution of the mean received powers, indicating that the MMSE receiver is robust to the randomness of the signatures and received powers.

Since the background noise V is proper complex Gaussian, it is clear that conditioned on $\boldsymbol{X}$, the distribution of $\mathcal{I}_2^{(N)}$ converges in probability to a proper complex Gaussian distribution. Also note that conditioned on $\boldsymbol{X}$, $\mathcal{I}_1^{(N)}$, and $\mathcal{I}_2^{(N)}$ are independent. Therefore, based on Theorem 4.1, we conclude that the conditional distribution of the overall receiver output $\mathcal{I}^{(N)}$, given $\boldsymbol{X}$, converges in probability to a proper complex Gaussian distribution. Heuristically, given the received signatures and received powers, the output interference in a large system is approximately Gaussian with high probability. This result is particularly useful for systems in which the powers vary at a slower rate than the information symbols and repetition of the same signatures is adopted. The reasoning is as follows. Assuming the information symbols are independent (which is valid when interleaving and de-interleaving are employed), the output interference, given the received signatures and received powers, is independent across symbol intervals, and the Gaussianity of the conditional distribution greatly simplifies the performance analysis and characterization of channel capacity.

We now take a networking perspective and proceed to characterize the network capacity of single-class systems.

Building on Theorem 4.1, we take the QoS requirement as meeting the SIR constraints. This is sensible because in view of the Gaussianity of the output interference, the SIR is of fundamental interest for detection and characterization of channel capacity, and is therefore the key parameter that governs the system performance. Also because any (positive) scaled version of the MMSE receiver results in the same SIR, it suffices to use any (positive) scaled version of the MMSE receiver. Thus, there is really no need for knowledge of the desired user's instantaneous received power for constructing the receiver. This implies that the MMSE receiver is robust to channel uncertainty, which suggests that the MMSE receiver is particularly useful in a multiple-antenna wireless system, where channel estimation becomes more difficult.

In a single-class system, the received powers of all the users are identically distributed. Let F denote the received power distribution and μ its mean. Assume that F is continuous. In what follows, we first characterize the asymptotic SIR, which serves as the basis for identifying the network capacity. Recall that

$$\text{SIR}_1^{(N)} = \frac{P_1\left(s_1^H M_I^{-1} s_1\right)^2}{s_1^H M_I^{-1} M_I' M_I^{-1} s_1}.$$

In the special case where the received powers are identically distributed, we have that $M_I = \mu S_1 S_1^H + \eta I$. For convenience, define

$$t_i^{(N)} \triangleq \sqrt{P_i}\, b_i s_1^H M_I^{-1} s_i, \qquad i = 2, \ldots, K.$$

Then

$$s_1^H M_I^{-1} M_I' M_I^{-1} s_1 = \sum_{i=2}^{K} \left|t_i^{(N)}\right|^2 + \eta s_1^H M_I^{-2} s_1.$$

Let G^* denote the limiting distribution of the eigenvalues of the random matrix $S_1 E_1 S_1^H$ (it has been shown in [13] that the empirical distribution of the eigenvalues of $S_1 E_1 S_1^H$ converges weakly to G^* with probability one; see Appendix B for more details). Using the same techniques as in the proof of [20, Theorem 3.1], it can be shown that

$$\sum_{i=2}^{K} \left|t_i^{(N)}\right|^2 \xrightarrow{P} \int_0^\infty \frac{\lambda}{(\lambda+\eta)^2}\, dG^*(\lambda)$$

and

$$s_1^H M_I^{-2} s_1 \xrightarrow{P} \int_0^\infty \frac{1}{(\lambda+\eta)^2}\, dG^*(\lambda).$$

Then it follows that

$$\text{SIR}_1^{(N)} \xrightarrow{P} P_1 \int_0^\infty \frac{1}{\lambda+\eta}\, dG^*(\lambda). \tag{6}$$

Combining the preceding equation with (20) in Appendix B, we obtain that $\text{SIR}_1^{(N)}$ (given P_1) converges in probability to $P_1\beta'$, where

$$\beta' = \frac{1}{2\mu\eta}\left[\sqrt{(\mu(\alpha-1)+\eta)^2 + 4\mu\eta} - \mu(\alpha-1) - \eta\right]. \tag{7}$$

We are now ready to present our main result on the network capacity of a single-class system with MMSE receivers.

Theorem 4.2: The asymptotic network capacity $\alpha_\infty(a)$ of a system with the MMSE receiver is

$$\alpha_\infty(a) = \frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu} - \frac{\eta\gamma}{F^{-1}(1-a)}.$$

Roughly speaking, in a large network, if the users choose their signatures randomly and independently, then up to

$$N \cdot \left(\frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu} - \frac{\eta\gamma}{F^{-1}(1-a)}\right)$$

users are admissible in the system.

It is of interest to compare the preceding result with the corresponding characterization of network capacity of a system with optimal deterministic signatures carried out in [19]. It is shown in [19] that the network capacity of a system with optimal signature allocation is $\frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu}$. Compared to Theorem 4.2, we conclude that by allocating signatures "intelligently," the network capacity is increased by $\frac{\eta\gamma}{F^{-1}(1-a)}$. However, as the signal-to-noise ratio (SNR) increases, the gap between the network capacity corresponding to optimal signatures and that corresponding to random signatures vanishes. That is, at the cost of transmission power, we can drive the gap between the network capacity corresponding to optimal signatures and that corresponding to random signatures arbitrarily small. This observation leads to the conclusion that from the viewpoint of network capacity, systems with MMSE receivers are *robust* to the choice of signature sets.

V. Proofs of Main Results

A. Technical Lemmas

First we collect a few results on the moments of the $t_i^{(N)}$'s, the proofs of which involve techniques similar to those in [20]. The tedious details can be found in [18]. (In what follows, we use the standard notation $\text{Re}\{z\}$, $\text{Im}\{z\}$, and $\bar{z}$ to denote the real part, imaginary part, and complex conjugate, respectively, of a complex number z.)

Lemma 5.1:

i)

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{i=2}^{K}\left|\text{Re}\left\{t_i^{(N)}\right\}\right|^2\right] = \frac{1}{2}\int_0^\infty \frac{\lambda}{(\lambda+\eta)^2}\, dG^*(\lambda).$$

ii)

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{i=2}^{K}\left|\text{Im}\left\{t_i^{(N)}\right\}\right|^2\right] = \frac{1}{2}\int_0^\infty \frac{\lambda}{(\lambda+\eta)^2}\, dG^*(\lambda).$$

iii)

$$\text{Var}\left(\sum_{i=2}^{K}\left|\text{Re}\left\{t_i^{(N)}\right\}\right|^2\right) \to 0.$$

iv)

$$\text{Var}\left(\sum_{i=2}^{K}\left|\text{Im}\left\{t_i^{(N)}\right\}\right|^2\right) \to 0.$$

v)

$$\mathbb{E}\left[\left|t_i^{(N)}\right|^4\right] \le \frac{C_1}{N^2}, \qquad i = 2, \ldots, K$$

where C_1 is a positive constant that does not depend on N.

For notational convenience, we let $Z^{(N)}$ denote the collection of the b_i's (excluding $i = 1$) when the degree of freedom is N, and $\boldsymbol{Z}$ the sequence $\{Z^{(N)}\}$. By our assumption, $\boldsymbol{X}$ and $\boldsymbol{Z}$ are independent. We need the following lemmas to prove part b) in Theorem 4.1, our main result on the conditional distribution of the output MAI. Lemmas 5.2 and 5.3 are proved in Appendix C.

Lemma 5.2: Suppose $\boldsymbol{X}$ and $\boldsymbol{Z}$ are independent. Let $f^{(N)}(\boldsymbol{X}, \boldsymbol{Z})$ be a Borel function, $N = 1, 2, \ldots$. If $\{L\}$ is a subsequence of $\{N\}$ and the sequence $\{f^{(L)}(\boldsymbol{X}, \boldsymbol{Z})\}$ converges in probability to some constant c_0, then there exists a further subsequence $\{L'\}$ of $\{L\}$ such that

$$P\left\{\omega\colon f^{(L')}(\boldsymbol{X}(\omega), \boldsymbol{Z}) \xrightarrow{P} c_0\right\} = 1.$$

Let

$$a_g = \frac{\sqrt{P_1}}{1 + P_1\int_0^\infty \frac{1}{\lambda+\eta}\, dG^*(\lambda)}.$$

Define for $N = 1, 2, \ldots$

$$U^{(N)}(\boldsymbol{X}, \boldsymbol{Z}) \triangleq \max_{2\le i\le K} \frac{t_i^{(N)}\sqrt{P_1}}{1 + P_1 s_1^H M_I^{-1} s_1}$$

$$W_1^{(N)}(\boldsymbol{X}, \boldsymbol{Z}) \triangleq \sum_{i=2}^{K} \frac{\left|\text{Re}\left\{t_i^{(N)}\right\}\right|^2 P_1}{(1 + P_1 s_1^H M_I^{-1} s_1)^2}$$

$$W_2^{(N)}(\boldsymbol{X}, \boldsymbol{Z}) \triangleq \sum_{i=2}^{K} \frac{\left|\text{Im}\left\{t_i^{(N)}\right\}\right|^2 P_1}{(1 + P_1 s_1^H M_I^{-1} s_1)^2}.$$

Lemma 5.3: Every subsequence $\{N'\}$ of $\{N\}$ contains a further subsequence $\{N''\}$ such that

$$P\left\{\omega\colon \left|U^{(N'')}(\boldsymbol{X}(\omega), \boldsymbol{Z})\right| \xrightarrow{P} 0\right\} = 1$$

$$P\left\{\omega\colon W_1^{(N'')}(\boldsymbol{X}(\omega), \boldsymbol{Z}) \xrightarrow{P} \frac{1}{2}a_g^2 \int_0^\infty \frac{\lambda}{(\lambda+\eta)^2}\, dG^*(\lambda)\right\} = 1$$

and

$$P\left\{\omega\colon W_2^{(N'')}(\boldsymbol{X}(\omega), \boldsymbol{Z}) \xrightarrow{P} \frac{1}{2}a_g^2 \int_0^\infty \frac{\lambda}{(\lambda+\eta)^2}\, dG^*(\lambda)\right\} = 1.$$

B. Proof of Theorem 4.1

Proof of Part a): Using a similar approach to the proof of [15, Lemma 4.3], it can be shown that

$$\frac{\sqrt{P_1}}{1+P_1 s_1^H M_I^{-1} s_1} \xrightarrow{P} a_g. \tag{8}$$

Then it suffices to show that $\sum_{i=2}^K t_i^{(N)}$ converges in distribution to a proper complex Gaussian random variable. To that end, based on Lemma A.1 in Appendix A, it remains to verify that the three conditions in Lemma A.1 are satisfied [20].

Using parts i) and ii) in Lemma 5.1, it is straightforward to see that

$$\mathbb{E}\left[\max_{2\le i\le K}\left|t_i^{(N)}\right|^2\right] \le \mathbb{E}\left[\sum_{i=2}^K \left|t_i^{(N)}\right|^2\right] \le \frac{1}{\eta}. \tag{9}$$

Fix $\epsilon > 0$. By exploiting part v) in Lemma 5.1, we have that

$$\begin{aligned} P\left\{\max_{2\le i\le K}\left|t_i^{(N)}\right| > \epsilon\right\} &\le \sum_{i=2}^K P\left\{\left|t_i^{(N)}\right| > \epsilon\right\} \\ &\le \sum_{i=2}^K \frac{\mathbb{E}\left[\left(t_i^{(N)}\right)^4\right]}{\epsilon^4} \\ &\le \frac{K}{\epsilon^4}\frac{C_1}{N^2} \\ &\to 0. \end{aligned} \tag{10}$$

Furthermore, combining parts i)–iv) of Lemma 5.1 with Chebyshev's inequality, we obtain that

$$\sum_{i=2}^K \left|\mathrm{Re}\left\{t_i^{(N)}\right\}\right|^2 \xrightarrow{P} \frac{1}{2}\int_0^\infty \frac{\lambda}{(\lambda+\eta)^2}\, dG^*(\lambda) \tag{11}$$

$$\sum_{i=2}^K \left|\mathrm{Im}\left\{t_i^{(N)}\right\}\right|^2 \xrightarrow{P} \frac{1}{2}\int_0^\infty \frac{\lambda}{(\lambda+\eta)^2}\, dG^*(\lambda) \tag{12}$$

thereby completing the proof of part a).

Proof of Part b): Recall that

$$\mathcal{I}_1^{(N)} = \sum_{i=2}^K \frac{\sqrt{P_1}t_i^{(N)}}{1+P_1 s_1^H M_I^{-1} s_1}.$$

It is clear that conditioned on the signatures and received powers, the array $\{t_i^{(N)}\}$ still forms a complex martingale difference array with respect to $\{\mathcal{F}_{N,i}\}$. In what follows, first we show that $\max_{2\le i\le K}|t_i^{(N)}|$ is bounded in L_2 norm for almost every realization of $\boldsymbol{X}$.

Because $M_I \succeq \eta I \succ 0$, we have that $0 \prec M_I^{-1} \preceq \frac{1}{\eta} I$. (By matrix inequality $A \succ B$ ($A \succeq B$), we mean that $A - B$ is positive definite (semidefinite).) Also by Assumption (3.P2), we assume that the P_i's are bounded above by d_1 and the μ_i's are bounded below by $d_2 > 0$. It then follows that

$$\begin{aligned} &\mathbb{E}\left[\left(\max_{2\le i\le K}\left|t_i^{(N)}\right|\right)^2\middle|\boldsymbol{X}\right] \\ &\le \mathbb{E}\left[s_1^H M_I^{-1}\left(\sum_{i=2}^K P_i s_i s_i^H\right)M_I^{-1}s_1\middle|\boldsymbol{X}\right] \\ &\le \frac{d_1}{d_2}\mathbb{E}\left[s_1^H M_I^{-1}\left(\sum_{i=2}^K \mu_i s_i s_i^H\right)M_I^{-1}s_1\middle|\boldsymbol{X}\right] \\ &\le \frac{d_1}{d_2} s_1^H M_I^{-1} s_1, \qquad \text{since } \eta s_1^H M_I^{-2} s_1 \ge 0 \\ &\le \frac{d_1}{d_2\eta}. \end{aligned}$$

Then it is straightforward to see that

$$\mathbb{E}\left[\left|U^{(N)}(\boldsymbol{X},\boldsymbol{Z})\right|^2\middle|\boldsymbol{X}\right] \le \frac{P_1}{\eta}.$$

By the definition of conditional expectation, $\mathbb{E}[|U^{(N)}(\boldsymbol{X},\boldsymbol{Z})|^2|\boldsymbol{X}]$ is a function of $\boldsymbol{X}$ only. Since $\boldsymbol{X}$ and $\boldsymbol{Z}$ are independent, it follows that for almost every $\omega\in\Omega$

$$\mathbb{E}\left[\left(U^{(N)}(\boldsymbol{X}(\omega),\boldsymbol{Z})\right)^2\right] \le \frac{P_1}{\eta}. \tag{13}$$

Next observe that for any subsequence $\{N'\}$ of $\{N\}$, by Lemma 5.3, there exists a further subsequence $\{N''\}$ such that

$$P\left\{\omega: \left|U^{(N'')}(\boldsymbol{X}(\omega),\boldsymbol{Z})\right| \xrightarrow{P} 0\right\} = 1$$

$$P\left\{\omega: W_1^{(N'')}(\boldsymbol{X}(\omega),\boldsymbol{Z}) \xrightarrow{P} \frac{1}{2}a_g^2\int_0^\infty \frac{\lambda}{(\lambda+\eta)^2}\, dG^*(\lambda)\right\} = 1$$

and

$$P\left\{\omega: W_2^{(N'')}(\boldsymbol{X}(\omega),\boldsymbol{Z}) \xrightarrow{P} \frac{1}{2}a_g^2\int_0^\infty \frac{\lambda}{(\lambda+\eta)^2}\, dG^*(\lambda)\right\} = 1.$$

Combining the above facts with (13), we conclude that there exists a set E such that $P(E) = 1$ and for any $\omega\in E$, the following three conditions are satisfied simultaneously.

1) The L_2 norm of $|U^{(N'')}(\boldsymbol{X}(\omega),\boldsymbol{Z})|$ is bounded for all N''.
2) $|U^{(N'')}(\boldsymbol{X}(\omega),\boldsymbol{Z})|$ converges to 0 in probability.
3) Both $W_1^{(N'')}(\boldsymbol{X}(\omega),\boldsymbol{Z})$ and $W_2^{(N'')}(\boldsymbol{X}(\omega),\boldsymbol{Z})$ converge in probability to $\frac{1}{2}a_g^2\int_0^\infty \frac{\lambda}{(\lambda+\eta)^2}\,dG^*(\lambda)$.

Hence, we apply Lemma A.1 in Appendix A to conclude that the conditional distribution of $\mathcal{I}_1^{(N'')}$, given $\boldsymbol{X}$, converges almost surely to a proper complex Gaussian distribution with zero mean and covariance $a_g^2\int_0^\infty \frac{\lambda}{(\lambda+\eta)^2}\,dG^*(\lambda)$. Therefore, by Definition 3.2, $P_N^{\boldsymbol{X}}$ converges in probability to a complex Gaussian distribution with mean 0 and covariance $a_g^2\int_0^\infty \frac{1}{\lambda+\eta}\,dG^*(\lambda)$. This completes the proof.

C. Proof of Theorem 4.2

For convenience, we define

$$T^{(N)} \triangleq \frac{(s_1^H M_I^{-1} s_1)^2}{s_1^H M_I^{-1} M_I' M_I^{-1} s_1}$$

where the degree of freedom is N (correspondingly, the s_k's are N-dimensional vectors). As shown in (6)

$$T^{(N)} \xrightarrow{P} \beta'$$

where β' is given in (7).

Let $\{N_s\}$ denote the subsequence of $\{N\}$ such that

$$\lim_{N_s\to\infty}\alpha_{N_s}(a) = \limsup_{N\to\infty}\alpha_N(a)$$

and $\{N_i\}$ denotes the subsequence of $\{N\}$ such that

$$\lim_{N_i\to\infty}\alpha_{N_i}(a) = \liminf_{N\to\infty}\alpha_N(a).$$

It is clear that

$$T^{(N_s)} \xrightarrow{P} \beta' \tag{14}$$

and

$$T^{(N_i)} \xrightarrow{P} \beta'. \tag{15}$$

In what follows, we first prove that $\alpha_\infty(a)$ is upper-bounded by

$$\frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu} - \frac{\eta\gamma}{F^{-1}(1-a)}.$$

Applying Lemma 3.1 to (14), we have that there exists a subsequence $\{N_{ss}\}$ of $\{N_s\}$ such that

$$T^{(N_{ss})} \xrightarrow{\text{a.s.}} \beta'.$$

Fix $\delta > 0$. By Egoroff's Theorem (cf. [19]), there exists a measurable set A_1 such that $P(A_1) < \delta$ and $T^{(N_{ss})}$ converges to β' uniformly on $\overline{A_1} = \Omega \setminus A_1$. Then, for fixed $\epsilon > 0$, there exists an integer $N_0(\epsilon)$ such that for all $N_{ss} \geq N_0(\epsilon)$, and every point in $\overline{A_1}$

$$\beta' - \epsilon \leq T^{(N_{ss})} \leq \beta' + \epsilon.$$

It follows that for all $N_{ss} \geq N_0(\epsilon)$

$$\begin{aligned} &P\left\{\text{SIR}_1^{(N_{ss})} \geq \gamma\right\} \\ &\quad = P\left\{\left(\text{SIR}_1^{(N_{ss})} \geq \gamma\right) \cap A\right\} + P\left\{\left(\text{SIR}_1^{(N_{ss})} \geq \gamma\right) \cap \overline{A}\right\} \\ &\quad \leq \delta + P\left\{(P_1(\beta' + \epsilon) \geq \gamma) \cap \overline{A}\right\}. \end{aligned}$$

Combining the above with the definition of network capacity yields that

$$a - \delta < 1 - F\left(\frac{\gamma}{\beta' + \epsilon}\right).$$

Based on the above inequality, it can be shown that

$$\begin{aligned} \alpha_{N_{ss}}(a) \leq{} & \frac{F^{-1}(1-a+\delta)}{(\gamma - \epsilon F^{-1}(1-a+\delta))\,\mu} \\ & + 1 - \frac{\eta}{\mu} - \eta\left(\frac{\gamma}{F^{-1}(1-a+\delta)} - \epsilon\right). \end{aligned}$$

Because both ϵ and δ are arbitrary positive numbers, we conclude that

$$\lim_{N_{ss}\to\infty} \alpha_{N_{ss}} \leq \frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu} - \frac{\eta\gamma}{F^{-1}(1-a)} \tag{16}$$

which dictates that

$$\begin{aligned} \limsup_{N\to\infty} \alpha_N(a) &= \lim_{N_{ss}\to\infty} \alpha_{N_{ss}} \\ &\leq \frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu} - \frac{\eta\gamma}{F^{-1}(1-a)}. \end{aligned}$$

It remains to show that $\alpha_\infty(a)$ is lower-bounded by

$$\frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu} - \frac{\eta\gamma}{F^{-1}(1-a)}.$$

We apply Lemma 3.1 to (15) to conclude that there exists a subsequence $\{N_{ii}\}$ of $\{N_i\}$ such that

$$T^{(N_{ii})} \xrightarrow{a.s.} \beta'.$$

For a fixed $\delta > 0$, we appeal to Egoroff's theorem again and conclude that there exists a measurable set A_2 such that $P(A_2) < \delta$ and $T^{(N_{ii})}$ converges to β' uniformly on $\overline{A_2} = \Omega \setminus A_1$. Then, for fixed $\epsilon > 0$, there exists an integer $N_1(\epsilon)$ such that for all $N_{ii} \geq N_0(\epsilon)$, and every point in $\overline{A_2}$

$$\beta' - \epsilon \leq T^{(N_{ii})} \leq \beta' + \epsilon.$$

It follows that for all $N_{ii} \geq N_0(\epsilon)$

$$\begin{aligned} P\left\{\text{SIR}_1^{(N_{ii})} \geq \gamma\right\} &\geq P\left\{(P_1(\beta' - \epsilon) \geq \gamma) \cap \overline{A}\right\} \\ &\geq 1 - F\left(\frac{\gamma}{\beta' - \epsilon}\right) - \delta. \end{aligned}$$

Then it can be shown that

$$\begin{aligned} \alpha_{N_{ii}}(a) \geq{} & \frac{F^{-1}(1-a-\delta)}{(\gamma + \epsilon F^{-1}(1-a-\delta))\mu} \\ & + 1 - \frac{\eta}{\mu} - \eta\left(\frac{\gamma}{F^{-1}(1-a-\delta)} + \epsilon\right). \end{aligned}$$

Therefore, we conclude that

$$\lim_{N_{ii}\to\infty} \alpha_{N_{ii}} \geq \frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu} - \frac{\eta\gamma}{F^{-1}(1-a)} \tag{17}$$

which leads to

$$\begin{aligned} \liminf_{N\to\infty} \alpha_N(a) &= \lim_{N_{ii}\to\infty} \alpha_{N_{ii}} \\ &\geq \frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu} - \frac{\eta\gamma}{F^{-1}(1-a)} \end{aligned}$$

thus completing the proof.

VI. Conclusion

We consider a canonical symbol-synchronous discrete-time model for wireless multiuser systems with MMSE receivers. We focus on the cases where the signatures are modeled as random and take values in complex space. First, we characterize the conditional distribution of the output MAI of the MMSE receiver. By appealing to conditional weak convergence, we have found that under the assumptions 3.P1)–3.P3), the conditional distribution of the output MAI, given the received signatures and received powers, converges in probability to a proper complex Gaussian distribution that does not depend on the signatures and depends weakly on the mean received powers. This result indicates that in a large system, the overall output interference of the MMSE receiver is approximately Gaussian with high probability, and the SIR is of fundamental interest.

Building on the Gaussianity of the output interference, we then take a networking perspective and identify the network capacity of single-class systems with random spreading. We have found that the network capacity can be expressed uniquely in terms of the SIR requirements and received power distributions. Compared to the network capacity corresponding to the optimal signature allocation characterized in [19], we conclude that at the cost of transmission power, the gap between the network capacity corresponding to optimal signatures and that corresponding to random signatures can be made arbitrarily small. This observation leads to the conclusion that from the viewpoint of network capacity, systems with MMSE receivers are robust to the randomness of signatures.

Our results are useful for performance analysis and characterization of system limits. In particular, the bit error probability (corresponding to a specific modulation scheme) can easily be obtained in terms of the SIR. Based on the calculation of the bit error probability, we can calculate packet error probability, which in turn impacts performance measures at the network layer, such as throughput and packet delay. Our result on the network capacity is also potentially useful for network-level resource allocation problems such as admission control and power control in a large system.

Appendix A
Complex Martingale Difference Array and Dependent Central Limit Theorem

In what follows, we generalize the dependent central limit theorem [8] to the complex case.

Recall $t_i^{(N)} = \sqrt{P_i}\, b_i s_1^H M_I^{-1} s_i$. It is straightforward to see that $\mathbb{E}[t_i^{(N)}] = 0$, and

$$\mathbb{E}\left[\left(t_i^{(N)}\right)^2\right] = \mathbb{E}[b_i^2]\mathbb{E}[P_i]\mathbb{E}\left[(s_1^H M_I^{-1} s_i)^2\right] = 0$$

which indicates that $t_i^{(N)}$ is proper.

Let $\mathcal{F}_{N,i}$ denote the σ-algebra generated by $\{t_2^{(N)}, \ldots, t_i^{(N)}\}$

$$\mathcal{F}_{N,i} \triangleq \sigma\left(t_2^{(N)}, \ldots, t_i^{(N)}\right), \qquad i = 2, \ldots, K.$$

That is,

$$\mathcal{F}_{N,i} = \sigma\left(\mathrm{Re}\left\{t_2^{(N)}\right\}, \mathrm{Im}\left\{t_2^{(N)}\right\}, \ldots, \mathrm{Re}\left\{t_i^{(N)}\right\}, \mathrm{Im}\left\{t_i^{(N)}\right\}\right).$$

It can be shown that for $i = 2, \ldots, K$

$$\mathbb{E}\left[\mathrm{Re}\left\{t_i^{(N)}\right\}\middle|\mathcal{F}_{N,i-1}\right] = 0. \tag{18}$$

Therefore, the array $\{\mathrm{Re}\{t_i^{(N)}\}, i = 2, \ldots, K\}$ is a *martingale difference array* with respect to $\{\mathcal{F}_{N,i}\}$ [8]. Similarly, we have that the array $\{\mathrm{Im}\{t_i^{(N)}\}, i = 2, \ldots, K\}$ is a martingale difference array with respect to $\{\mathcal{F}_{N,i}\}$. Then it follows that

$$\mathbb{E}\left[t_i^{(N)}\middle|\mathcal{F}_{N,i-1}\right] = 0. \tag{19}$$

We call the array $\{t_i^{(N)}, i = 2, \ldots, K\}$ a *complex martingale difference array* with respect to $\{\mathcal{F}_{N,i}\}$.

The proofs of our main results make use of the following lemma.

Lemma A.1: Suppose the $t_i^{(N)}$'s are proper complex random variables, and the array $\{t_i^{(N)}, i = 2, \ldots, K\}$ is a complex martingale difference array with respect to $\{\mathcal{F}_{N,i}\}$. Suppose the following conditions hold:

M1) $\max_{2\le i\le K} |t_i^{(N)}|$ is bounded in L_2 norm;

M2) $\max_{2\le i\le K} |t_i^{(N)}|$ converges in probability to 0 as $N \to \infty$;

M3) Both

$$\sum_{i=2}^{K} |\mathrm{Re}\{t_i^{(N)}\}|^2 \quad \text{and} \quad \sum_{i=2}^{K} |\mathrm{Im}\{t_i^{(N)}\}|^2$$

converge in probability to a constant a_0 as $N \to \infty$.

Then $\sum_{i=2}^{K} t_i^{(N)}$ converges in distribution to a proper complex Gaussian random variable with zero mean and covariance $2a_0$.

Proof: Because the $t_i^{(N)}$'s are pairwise uncorrelated, it follows that

$$\mathbb{E}\left[\left(\sum_{i=2}^{K} t_i^{(N)}\right)^2\right] = \sum_{i=2}^{K} \mathbb{E}\left[\left(t_i^{(N)}\right)^2\right] = 0,$$

which implies that $\sum_{i=2}^{K} t_i^{(N)}$ is proper. Therefore, it suffices to show that $\mathrm{Re}\{\sum_{i=2}^{K} t_i^{(N)}\}$ and $\mathrm{Im}\{\sum_{i=2}^{K} t_i^{(N)}\}$ converge in distribution to a real Gaussian random variable with zero mean and variance a_0. To this end, we appeal to [8]. In what follows, we verify that the corresponding conditions are satisfied.

Because

$$\max_{2\le i\le K} |\mathrm{Re}\{t_i^{(N)}\}| \le \max_{2\le i\le K} |t_i^{(N)}|$$

Condition M1) implies that $\max_{2\le i\le K} |\mathrm{Re}\{t_i^{(N)}\}|$ is bounded in L_2 norm. Moreover, we have that

$$P\left\{\max_{2\le i\le K}\left|\mathrm{Re}\left\{t_i^{(N)}\right\}\right| \ge \epsilon\right\} \le P\left\{\max_{2\le i\le K}\left|t_i^{(N)}\right| \ge \epsilon\right\}.$$

Based on Condition M2), it follows that $\max_{2\le i\le K} |\mathrm{Re}\{t_i^{(N)}\}|$ converges in probability to 0. Combining the above with Condition M3), we appeal to [8] and conclude that $\mathrm{Re}\{\sum_{i=2}^{K} t_i^{(N)}\}$ converges in distribution to a real Gaussian random variable with zero mean and variance a_0. Along the same lines, it can be shown that $\mathrm{Im}\{\sum_{i=2}^{K} t_i^{(N)}\}$ also converges in distribution to a real Gaussian random variable with zero mean and variance a_0, thereby completing the proof. □

APPENDIX B
A RANDOM MATRIX THEOREM

Denote the eigenvalues of the random matrix $S_1 E_1 S_1^H$ by $\lambda_1, \ldots, \lambda_N$ (they are random, depending on the realization of S_1), and the empirical distribution of the eigenvalues by G_N. It is shown in [13, Theorem 1.1] that if Condition 3.P1) holds, then G_N converges weakly to a distribution function G^* with probability one, and the Stieltjes transform $m(z)$ of G^* is the solution of the following functional equation:

$$m(z) = \frac{1}{-z + \alpha \int \frac{\mu}{1+\mu m(z)}\, dH_\mu(\mu)} \tag{20}$$

where the Stieltjes transform $m(z)$ of any distribution G is defined as

$$m_G(z) \triangleq \int \frac{1}{\lambda - z}\, dG(\lambda)$$

for $z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C}, \mathrm{Im}\{z\} > 0\}$.

APPENDIX C
PROOFS OF TECHNICAL LEMMAS

A. *Proof of Lemma 5.2*

Because $f^{(L)}(\boldsymbol{X}, \boldsymbol{Z}) \xrightarrow{P} c_0$, by Lemma 3.1 there exists a subsequence $\{L'\}$, which is a further subsequence of some subsequence of $\{L\}$, such that

$$P\left\{f^{(L')}(\boldsymbol{X}, \boldsymbol{Z}) \longrightarrow c_0\right\} = 1.$$

That is,

$$I_{\{f^{(L')}(\boldsymbol{X},\boldsymbol{Z})\longrightarrow c_0\}} = 1 \qquad \text{a.s.}$$

where I_A is the indicator function of the set A. Because $\boldsymbol{X}$ and $\boldsymbol{Z}$ are independent, it follows that

$$P\left\{\omega\colon P\left\{\omega'\colon f^{(L')}(\boldsymbol{X}(\omega), \boldsymbol{Z}(\omega')) \longrightarrow c_0\right\} = 1\right\} = 1$$

which implies that

$$P\left\{\omega\colon f^{(L')}(\boldsymbol{X}(\omega), \boldsymbol{Z}) \xrightarrow{P} c_0\right\} = 1$$

completing the proof.

B. Proof of Lemma 5.3

It is clear that for any given S, M_I is positive definite, and hence M_I^{-1} is also positive definite. It then follows that for every S

$$0 < \frac{\sqrt{P_1}}{1 + P_1 s_1^H M_I^{-1} s_1} < \sqrt{P_1}. \tag{21}$$

Combining (11) and (12) with (21), we have that

$$W_1^{(N)}(\boldsymbol{X}, \boldsymbol{Z}) \xrightarrow{P} \frac{1}{2} a_g^2 \int_0^\infty \frac{\lambda}{(\lambda+\eta)^2} dG^*(\lambda) \tag{22}$$

$$W_2^{(N)}(\boldsymbol{X}, \boldsymbol{Z}) \xrightarrow{P} \frac{1}{2} a_g^2 \int_0^\infty \frac{\lambda}{(\lambda+\eta)^2} dG^*(\lambda). \tag{23}$$

Then, for every subsequence $\{N'\}$ of $\{N\}$, combining (21) with (10) yields that

$$\left| U^{(N')}(\boldsymbol{X}, \boldsymbol{Z}) \right| \xrightarrow{P} 0.$$

Appealing to Lemma 5.2, we conclude that there exists a subsequence $\{J'\}$ of $\{N'\}$ such that

$$P\left\{\omega\colon \left| U^{(J')}(\boldsymbol{X}(\omega), \boldsymbol{Z}) \right| \xrightarrow{P} 0 \right\} = 1.$$

Based on (22) and (23), for the subsequence $\{J'\}$, we resort to Lemma 5.2 again and conclude that there exists a further subsequence $\{N''\}$ of $\{J'\}$ such that

$$P\left\{\omega\colon W_1^{(N'')}(\boldsymbol{X}(\omega), \boldsymbol{Z}) \xrightarrow{P} \frac{1}{2} a_g^2 \int_0^\infty \frac{\lambda}{(\lambda+\eta)^2} dG^*(\lambda) \right\} = 1$$

$$P\left\{\omega\colon W_2^{(N'')}(\boldsymbol{X}(\omega), \boldsymbol{Z}) \xrightarrow{P} \frac{1}{2} a_g^2 \int_0^\infty \frac{\lambda}{(\lambda+\eta)^2} dG^*(\lambda) \right\} = 1.$$

Furthermore, it is clear that

$$P\left\{\omega\colon \left| U^{(N'')}(\boldsymbol{X}(\omega), \boldsymbol{Z}) \right| \xrightarrow{P} 0 \right\} = 1$$

thereby concluding the proof.

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Improvement of Ashikhmin–Litsyn–Tsfasman Bound for Quantum Codes

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***Abstract*—We improve performance of the asymptotically good quantum codes constructed by Ashikhmin, Litsyn, and Tsfasman, by using more rational points on algebraic curves.**

***Index Terms*—Algebraic-geometry code, Ashikhmin–Litsyn–Tsfasman bound, quantum code.**

I. Introduction

Recently, quantum computation and quantum communication have attracted much attention, because the use of quantum-mechanical phenomena can offer unusual efficiency in computation and com-

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