CDMA Systems in Fading Channels: Admissibility, Network Capacity, and Power Control

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***Abstract*—We study the *admissibility* and *network capacity* of imperfect power-controlled Code-Division Multiple Access (CDMA) systems with linear receivers in fading environments. In a CDMA system, a set of users is *admissible* if their simultaneous transmission does not result in violation of any of their Quality-of-Service (QoS) requirements; the *network capacity* is the maximum number of admissible users. We consider a single-cell imperfect power-controlled CDMA system, assuming known received power distributions. We identify the network capacities of single-class systems with matched-filter (MF) receivers for both the deterministic and random signature cases. We also characterize the network capacity of single-class systems with linear Minimum-Mean-Square-Error (MMSE) receivers for the deterministic signature case. The network capacities can be expressed uniquely in terms of the users' signal-to-interference ratio (SIR) requirements and received power distributions. For multiple-class systems equipped with MF receivers, we find a necessary and sufficient condition on the admissibility for the random signature case, but only a sufficient condition for the deterministic signature case. We also introduce the notions of *effective target SIR* and *effective bandwidth*, which are useful in determining the admissibility and hence network capacity of an imperfect power-controlled system.**

***Index Terms*—Admissibility, CDMA, deterministic signature, fading channel, matched filter, MMSE, network capacity, power control, random signature, scale family.**

I. Introduction

THE last fifteen years have witnessed a tremendous growth of wireless networks. Due to the fast-growing demand for network capacity in wireless networks, it is essential to utilize efficiently the limited resources. The characterization of network capacity is therefore a fundamental and pressing issue in wireless network research. In this paper, we consider a model for the uplink of a single-cell symbol-synchronous Code-Division Multiple Access (CDMA) system in fading channels. The network therein consists of numerous mobile subscribers communicating with one base station, which is typically interconnected to a backbone network via a wired infrastructure.

Two approaches have been studied extensively to achieve efficient utilization of network resources in CDMA systems: *multiuser detection* and *power control*. Multiuser detection refers to the process of demodulating one or more user data streams from a nonorthogonal multiplex and is concerned with designing good receivers to achieve efficient interference suppression. In particular, among multiuser receivers, linear receivers have attracted a large amount of attention because they are practically appealing (see, e.g., [13]–[15], [30]). Power control, on the other hand, is implemented at the transmitter and is concerned with allocating powers to fulfill individual users' Quality-of-Service (QoS) requirements (see, e.g., [7], [9], [28], [36]). Since both linear receivers and power control are employed to suppress interference effectively and utilize network resources efficiently, it is natural to ask how linear receivers perform in power-controlled systems. In [27], Tse and Hanly characterized the network capacities for several important linear receivers via a notion of *effective bandwidth*, assuming users have random signatures. (See also earlier work in [9].) Viswanath, Anantharam, and Tse [31] studied the joint optimization problem of signature allocation and power control, and obtained simple characterizations of the network capacities of single-cell systems with linear Minimum-Mean-Square-Error (MMSE) receivers. However, both [27] and [31] assumed perfect power control in characterizing the network capacity. In a practical wireless communication system, due to delays and errors in power control and time-varying channel conditions, it is difficult to implement perfect power control, and the received powers typically fluctuate around the desired levels. Therefore, it is more appropriate to model the received powers as random. However, little work has been done on characterizing network capacity of imperfect power-controlled CDMA systems with linear receivers in fading channels.

In this paper, we focus primarily on the admissibility and network capacity of *imperfect* power-controlled CDMA systems with *matched filter* (MF) receivers in fading channels. Throughout this paper, we assume MF receivers unless specified otherwise. Each user in the system is assigned a signature onto which the user's information symbols are spread. Every user also has a minimum signal-to-interference (SIR) requirement. Roughly speaking, a set of users is said to be *admissible* if their simultaneous transmission does not result in violation of any of their SIR requirements; the *network capacity* is the maximum number of admissible users. Following the approach of [27], [30], we formulate the problem in an asymptotic setting in which we allow the number of users K and the degrees of freedom N (length of the signatures) to grow, while keeping their ratio fixed. The results are stated in terms of this ratio of number of users per degree of freedom. A feature that distinguishes this work from [27] is that the received power for

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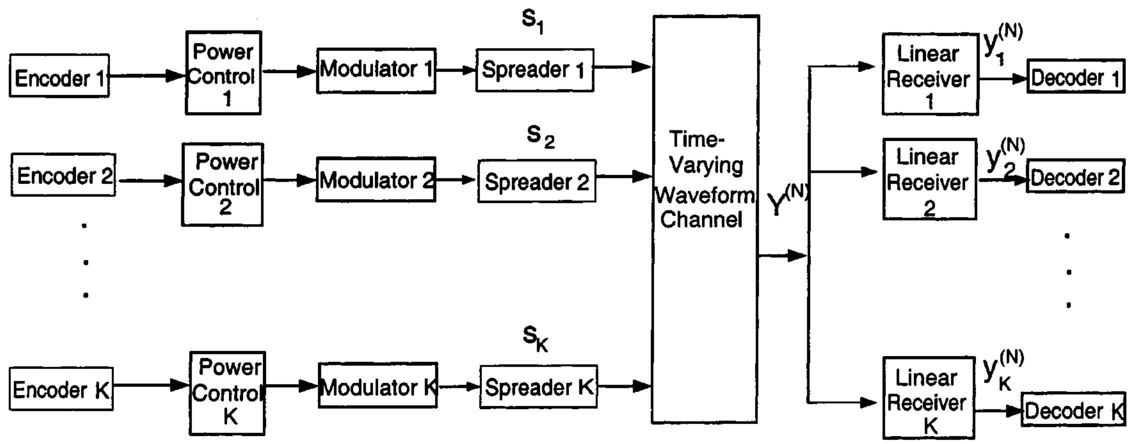

Fig. 1. A simplified block diagram of the uplink of a CDMA system with linear receivers.

each user in our model is random. The SIR requirements in our setting are also therefore probabilistic, unlike those of [27].

We treat separately systems with single class of users and systems with multiple classes of users. In each case, we consider both deterministic and random signatures. For a single-class systems with MF receivers, we identify the network capacities for both deterministic and random signature cases. We also characterize the network capacity of a single-class system with linear MMSE receivers for the deterministic signature case, which turns out to be exactly the same as that of the corresponding system with MF receivers. For multiple-class systems with MF receivers, we provide a necessary and sufficient condition for the random signature case, but only a sufficient condition for the deterministic signature case, for a set of users per degree of freedom to be admissible. The analysis in the deterministic signature case involves Welch-bound-equality (WBE) signature sets and inequalities, using results of [16], [34], as in [31]. Based on the results on the admissibility, we introduce the notions of *effective target SIR* and *effective bandwidth*, which are useful in determining the admissibility and hence network capacity of an imperfect power-controlled system.

The organization of the rest of this paper is as follows. In the next section, we provide the model description of an imperfect power-controlled CDMA system in fading channels. In Section III, we study the admissibility and identify the network capacities of single-class systems with MF receivers. We also identify the network capacity of single-class systems with linear MMSE receivers for the deterministic signature case. Then we address some issues on power control and introduce the notion of *effective target SIR*. In Section IV, we extend the study to multiple-class systems, and explore the notion of *effective bandwidth* for imperfect power-controlled CDMA systems. Finally, we draw some conclusions in Section V.

II. System Model

Consider a K-user CDMA system where the users transmit data over a fading channel. Fig. 1 depicts a simplified block diagram of the uplink of a CDMA system with linear receivers. Based on [27], [28], [30], we have the following discrete-time model for the uplink of a synchronous CDMA system: for any symbol interval, the baseband received signal at the front end of the receiver is

$$Y = \sum_{k=1}^{K} \sqrt{P_k^{(N)}} b_k s_k + V \tag{1}$$

where s_k, b_k, and $P_k^{(N)}$ are user k's signature, information symbol, and received power, respectively, $k = 1, \cdots, K$. The vector V is $\mathcal{N}(0, \eta I)$ background noise that comes from the sampling of the ambient white Gaussian noise with positive variance η. Note that different processing gains correspond to systems with different bandwidths, in which users may have different received powers. To keep our model more general, we use $P_k^{(N)}$ to denote user k's received power to emphasize the dependence of the received power on the processing gain N. Since our purpose is not to evaluate or optimize modulation performance, for simplicity, we assume that the modulation is antipodal, i.e., $b_k \in \{-1, 1\}$. This assumption is not crucial, but simplifies the analysis.

The signatures provide diversity gain and the corresponding model is of considerable interest. Following [27], [29], we assume that when the processing gain is N, s_k takes value in $\mathbb{R}^N$, $k = 1, \cdots, K$. We study the admissibility and network capacity for the following two cases:

- **The deterministic signature case:** In this case, the users' signatures are jointly designed and are deterministic (see, e.g., [16], [23], [31]). Our objective for the deterministic signature case is to find the tightest upper bound on the number of admissible users over all possible choices of signatures regardless of practical constraints.
- **The random signature case:** In this case, the users choose their signatures randomly and independently, which is applicable to several practical scenarios [15], [27], [30] (we will elaborate further on this point in Section III). Our objective for the random signature case is to characterize the number of admissible users in this more realistic setting.

We assume that each user is capable of decentralized power control. One common approach to implementing power control in practical CDMA systems is to drive the received powers for all the users having the same QoS requirements to be a fixed predetermined value all the time, namely, *power balancing* (see, e.g., [7], [32]). In fact, as illustrated in various chapters in [29], equal received power is the optimal power allocation for users having same QoS requirements in most modulation/demodulation systems. Because of fading effects and power control errors, however, the received powers typically fluctuate around the desired levels in a practical system. Accordingly, we assume that the received powers are random (due to imperfect power control), and that the received powers across different users are independent. Throughout this paper, we also assume that power control is good enough to ensure that the fluctuation of received powers around their expectations is uniformly bounded (say by d) with probability one, which is reasonable in any practical system.

Let D denote the support of the received powers in the limiting regime (as $N \to \infty$). Define $D^+ \stackrel{\triangle}{=} D \cap (0, \infty)$. For technical reasons, we assume that D^+ is connected. (By [22, Proposition 4.12], D^+ can be either an interval or a single point.) We assume throughout that the distributions of the received powers are known. This assumption is reasonable because the distributions of the received powers can usually be obtained through measurements (see, e.g., [21]).

Typically, since the output interference in a large CDMA system can be approximated as Gaussian (see, e.g., [30]), it is reasonable to take the QoS requirement as meeting the SIR constraint (see, e.g., [36], [27], [30]). We define the SIR to be the ratio of the desired signal power to the sum of the noise and multiple-access interference (MAI) powers at the output of the receiver in a symbol interval [14]. Because of the randomness of the received powers and/or signatures, the SIR is random as well. Therefore, we adopt a probabilistic model for the users' QoS requirements as follows (cf., [18], [19]):

$$P\left\{\mathrm{SIR}_k^{(N)} \geq \gamma_k\right\} > a_k$$

where $\mathrm{SIR}_k^{(N)}$ is the achieved SIR of user k when the processing gain is N, γ_k the *target SIR* of user k, and $a_k \in [0,1), k = 1, \cdots, K$.

Our results are asymptotic in nature, with both K and N going to infinity. As is standard (see, e.g., [27], [30]), $\alpha \stackrel{\triangle}{=} K/N$ is taken to be fixed as N goes to infinity. In fact, we only need $\lim_{N\to\infty} K/N = \alpha$; however, it is more convenient to fix this ratio in the following discussions.

III. Single-Class Systems

Typically, fading channel gains are assumed to be stationary and ergodic (see, e.g., [8], [11]), and all the users are assumed to have independent and identically distributed (i.i.d.) channel gains (see, e.g., [11]). Accordingly, we model the received powers for all users in one class (a set of users having the same QoS requirements) to be independent and identically distributed in a large network.

In this section, we consider a single-class system. We use $F^{(N)}$ to denote the received power distribution when the processing gain is N, and F the received power distribution in the limiting regime (as $N \to \infty$). We assume that the $F^{(N)}$'s and F are continuous on D^+, and that as we scale up the system, $F^{(N)}$ converges pointwise to F. We let $\mu^{(N)}$ and μ denote the expectations corresponding to $F^{(N)}$ and F, respectively.

The SIR achieved by the matched-filter receiver is

$$\mathrm{SIR}_k^{(N)} = \frac{P_k^{(N)}}{\eta + \sum\limits_{i \neq k} P_i^{(N)} \left(s_k^t s_i\right)^2}, \qquad k = 1, \cdots, K. \tag{2}$$

Because all the users have the same QoS requirements, we adopt the following probabilistic model for their SIR requirements:

$$P\left\{\mathrm{SIR}_k^{(N)} \geq \gamma\right\} > a, \qquad k = 1, \cdots, K$$

where $a \in [0, 1)$. We assume that $a \leq 1 - F(\eta\gamma)$, because otherwise it is impossible to meet the users' SIR requirements even when the output MAI vanishes.

A. Admissibility and Network Capacity

1) The Deterministic Signature Case: In this case, the users' signatures are deterministic, and the signatures of all the K users form a signature set when the processing gain is N. As $N \to \infty$, we have a sequence of signature sets. Let $\mathcal{F}$ denote the collection of sequences of signature sets satisfying the following condition:

$$\sup_{\substack{i,j \leq K \\ i \neq j}} \left(s_i^t s_j\right)^2 \log N \to 0, \qquad \text{as } N \to \infty \tag{3}$$

where s_i and s_j are the signatures for user i and j when the processing gain is $N, i, j \in \{1, \cdots, K\}$. Roughly speaking, the condition given in (3) requires that the crosscorrelation of any two users' signatures goes to zero at a rate faster than $1/\sqrt{\log N}$ as $N \to \infty$.

We observe that the condition in (3) is a very *weak* regularity condition on signature sets since the rate at which $1/\sqrt{\log N}$ goes to 0 is very slow. Because low crosscorrelation is essential for taking advantage of statistical multiplexing in CDMA systems (see, e.g., [10], [34]), we expect that $\mathcal{F}$ contains a large collection of interesting sequences of signature sets. For example, suppose the signatures are binary random spreading sequences. Then we claim that for any $i \neq j$

$$\left(s_i^t s_j\right)^2 - \frac{1}{N} \stackrel{\text{a.s.}}{\to} 0. \tag{4}$$

That is, for almost every realization, the crosscorrelation of any chosen two users' signatures goes to zero at the rate of $1/N$, which is considerably faster than $1/\sqrt{\log N}$. The above claim can easily be shown as follows: for any $i, j \in \{1, \cdots, K\}, i \neq j$

$$\left(s_i^t s_j\right)^2 = s_i^t s_j s_j^t s_i.$$

The matrix $s_j s_j^t$ is of rank 1. Clearly, the spectral radius of $s_j s_j^t$ is 1 for all N. Appealing to [26, Lemma 3.1], we have that

$$E\left[\left|s_i^t s_j s_j^t s_i - \frac{1}{N}\text{trace}\left(s_j s_j^t\right)\right|^6\right] \le \frac{c_0}{N^3}$$

where c_0 is some constant. Since $\text{trace}\,(s_j s_j^t) = 1$, we use the first Borel–Cantelli Lemma [2] to conclude that (4) holds, thereby verifying our claim.

We also note that, in general, the condition in (3) cannot be weakened further because otherwise the output MAI power may diverge (see [5]). In what follows, we confine ourselves to sequences in $\mathcal{F}$.

For convenience, let $\mathcal{X}_N$ denote the signature set $\{s_1, \cdots, s_K\}$. We define admissibility for a class of users in the deterministic signature case as follows: when the processing gain is N, α users per degree of freedom of the processing gain are *admissible* in the system if there exists a signature set $\mathcal{X}_N$ such that when the signatures in $\mathcal{X}_N$ are assigned to the users, the users' SIR requirements are satisfied, that is,

$$P\left\{\text{SIR}_k^{(N)} \ge \gamma\right\} > a, \qquad k = 1, \cdots, K$$

which is equivalent to

$$\sup_{\mathcal{X}_N} \min_k P\left\{\text{SIR}_k^{(N)} \ge \gamma\right\} > a. \tag{5}$$

Of particular interest is the maximum number of users admissible by the system, which is defined to be the *network capacity*, denoted by $\alpha_N(a)$, in terms of the number of users per degree of freedom of the processing gain, that is,

$$\alpha_N(a) \triangleq \sup\left\{\alpha \;\middle|\; \sup_{\mathcal{X}_N}\min_k P\left\{\text{SIR}_k^{(N)} \ge \gamma\right\} > a\right\}. \tag{6}$$

The *asymptotic network capacity* $\alpha_\infty(a)$ is defined as the limit of the sequence of network capacities $\{\alpha_N(a)\}$:

$$\alpha_\infty(a) \triangleq \lim_{N\to\infty} \alpha_N(a).$$

(We shall show in the proof of Theorem 3.1 that the limit of $\{\alpha_N(a)\}$ exists.)

Given a distribution function H, for $y \in [0, 1)$, we define (cf., [3, p. 53])

$$H^{-1}(y) \triangleq \inf\{x \,|\, H(x) > y\}.$$

Our main result on the network capacity of a system with MF receivers for the deterministic signature case is essentially as follows (the formal statement is given in Theorem 3.1).

Deterministic signature case, MF: The asymptotic network capacity is

$$\alpha_\infty(a) = \frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu}.$$

We note that in the special case of perfect power control, the asymptotic network capacity is simply $\frac{1}{\gamma} + 1 - \frac{\eta}{\mu}$, which is as given in [31].

As has been observed in [31], under perfect power control, the network capacity of a system with linear MMSE receivers is exactly the same as that of the corresponding system with MF receivers. Then a natural question to ask is, "Is the network capacity of a system with linear MMSE receivers still identical to that of the corresponding system with MF receivers when the power control is imperfect?" To answer this question, we first need to construct the linear MMSE receiver in an imperfect power-controlled system.

We assume that the MMSE receiver for user k has knowledge of user k's instantaneous received power $P_k^{(N)}$, which can be obtained through channel estimation. Because the received powers may vary from symbol to symbol due to imperfect power control, it is difficult to obtain knowledge of the other users' instantaneous received powers. As noted before, however, the distributions of the received powers can usually be obtained through measurements. Therefore, we assume that the MMSE receivers for each user has knowledge of the other users' mean received powers. In particular, the mean received powers are $\mu^{(N)}$ in the single-class case.

The linear MMSE receiver for user k generates an output of the form of $c_k^t Y^{(N)}$, where c_k is chosen to minimize the mean-square error

$$J_k = E\left[\left(c_k^t Y^{(N)} - b_k\right)^2\right].$$

It is easy to see that

$$c_k = \frac{\sqrt{P_k^{(N)}} M_{I,k}^{-1} s_k}{1 + P_k^{(N)} s_k^t M_{I,k}^{-1} s_k}$$

and the SIR achieved by the MMSE receiver for user k is (cf., [14], [27])

$$\text{SIR}_{k,\text{ms}}^{(N)} = \frac{P_k^{(N)} (s_k^t M_{I,k}^{-1} s_k)^2}{s_k^t M_{I,k}^{-1} M_{I,k}' M_{I,k}^{-1} s_k} \tag{7}$$

where

$$\begin{aligned} S_k &\triangleq [s_1, \cdots, s_{k-1}, s_{k+1}, \cdots, s_K] \\ D_k &\triangleq \text{diag}\left(P_1^{(N)}, \cdots, P_{k-1}^{(N)}, P_{k+1}^{(N)}, \cdots, P_K^{(N)}\right) \\ M_{I,k} &\triangleq \mu^{(N)} S_k S_k^t + \eta I \\ M_{I,k}' &\triangleq S_k D_k S_k^t + \eta I. \end{aligned}$$

Along the same lines as in the MF receiver case, we define the network capacity of a system with linear MMSE receivers as the maximum number of users admissible by the system, that is,

$$\alpha_{N,\text{ms}}(a) \triangleq \sup\left\{\alpha \;\middle|\; \sup_{\mathcal{X}_N}\min_k P\left\{\text{SIR}_{k,\text{ms}}^{(N)} \ge \gamma\right\} > a\right\}. \tag{8}$$

The *asymptotic network capacity* $\alpha_{\infty,\text{ms}}(a)$ is defined as

$$\alpha_{\infty,\text{ms}}(a) \triangleq \lim_{N\to\infty} \alpha_{N,\text{ms}}(a).$$

Our main result on the network capacity of a system with MMSE receivers for the deterministic signature case is essentially as follows (the formal statement is given in Theorem 3.1).

Deterministic signature case, MMSE: The asymptotic network capacity is

$$\alpha_{\infty,\text{ms}}(a) = \frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu}.$$

As would be expected, the asymptotic network capacity in the MMSE case is identical to that of the MF case. This observation is a generalization of a result in [31].

The proofs of our results on the asymptotic network capacity make heavy use of WBE signature sets (see [23] for a survey). For ease of reference, we restate the definition of WBE signature sets as follows [16].

Definition 1: Suppose $K \geq N$. A *WBE signature set* is a set of vectors (of length N) $\{s_1, \cdots, s_K\}$ with unit norm satisfying the following equality:

$$\sum_{i=1}^{K} \sum_{j=1}^{K} (s_i^t s_j)^2 = \frac{K^2}{N}.$$

An important property of WBE signature sets is $SS^t = (K/N)I$, where $S \triangleq [s_1, \cdots, s_K]$.

The achievability of the asymptotic network capacity, in both the MF and linear MMSE receiver cases, relies on the existence of WBE signature sets for each $K \geq N$. For signatures whose components are real, the existence and construction of WBE signature sets for each $K \geq N$ is provided in [31]. In practical communication systems, signature sets are usually constructed from linear codes. An interesting construction method of WBE signature sets from linear codes for each $K \geq N$ can be found in [23].

Under the choice of a WBE signature set, it is easy to show that

$$c_k = \frac{\sqrt{P_k^{(N)}} s_k}{\mu^{(N)}(\alpha - 1) + \eta + P_k^{(N)}}.$$

That is, in an imperfect power-controlled system, when the users' signatures form a WBE signature set, the MMSE receiver is just a scaled version of the MF receiver (cf., [31]), which reveals the underlying reason why the network capacity of a system with MMSE receivers is the same as that of the corresponding system with MF receivers.

To achieve the asymptotic network capacity, we need to find sequences of WBE signature sets in $\mathcal{F}$. It has also been shown in [23] that there exist large collections of WBE signature sets whose maximum crosscorrelations are equal to or only slightly larger than

$$\sqrt{\frac{1}{K-1}\left(\frac{K}{N} - 1\right)}$$

(which is the Welch lower bound on the maximum crosscorrelation of a signature set), for example, WBE signature sets corresponding to *small Kasami codes* or *Kerdock codes*. It is straightforward to see that the condition in (3) is satisfied by sequences of such WBE signature sets, which approach this lower bound asymptotically. In Appendix B we give two examples of sequences of WBE signature sets by exploiting some known results on Hamming codes and Bose–Chaudhuri–Hocquenghem (BCH) codes. These sequences of WBE signature sets easily satisfy the condition in (3) (i.e., they are in $\mathcal{F}$). However, for general $\alpha \geq 1$, the existence of sequences of WBE signature sets in $\mathcal{F}$ is open.

We are now ready to state formally our results on the asymptotic network capacity for the deterministic signature case.

Theorem 3.1 (Deterministic Signature Case): The asymptotic network capacity of a system with MF receivers satisfies

$$\alpha_\infty(a) \leq \frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu}.$$

Moreover, if there exists a sequence of WBE signature sets in $\mathcal{F}$ corresponding to some α ($\alpha \equiv K/N$) less than the right-hand side of the above inequality, then $\alpha_\infty(a) \geq \alpha$. The same result holds for the asymptotic network capacity $\alpha_{\infty,\mathrm{ms}}(a)$ of a system with linear MMSE receivers.

Roughly speaking, by Theorem 3.1, in a large system the network capacity can be approximated by the quantity $\frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu}$. Assuming the existence of WBE signature sets with "low" crosscorrelations, this network capacity is achieved by such a signature set. To be precise, if for any $\alpha < \frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu}$, there exists a sequence of WBE signature sets in $\mathcal{F}$ with $K/N = \alpha$, then the asymptotic network capacity is exactly

$$\alpha_\infty(a) = \frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu}.$$

The same holds for the MMSE receiver case. Because the proof of Theorem 3.1 is rather technical, we defer the details to Section III-C.

In a practical wireless system, it is often more reasonable to assume that the signatures are randomly and independently chosen because of the time-varying channel distortion, arbitrary time delays, etc. Therefore, it is of considerable interest to characterize the network capacity of a system with random signatures, which we consider in the next subsection.

2) The Random Signature Case: In this case, the users choose their signatures randomly and independently. The model for random signatures is as follows: $s_k = \frac{1}{\sqrt{N}}(s_{k1}, \cdots, s_{kN})^t$, where the s_{ki}'s are i.i.d. with mean zero and variance 1, $i = 1, \cdots, N$ and $k = 1, \cdots, K$. For technical reasons, we assume that $E[s_{11}^4]$ is finite. The random signature model is applicable to several practical systems, for example, systems employing long pseudo-random spreading sequences, and systems where the signatures are picked randomly and independently when the users are admitted into the system initially [15], [27], [30].

We assume that the received powers and the signatures are independent, which is valid in many practical scenarios, for example, in a system employing long pseudo-random spreading sequences.

Because the received powers are i.i.d. and the signatures are i.i.d., the users' SIR's are identically distributed. Hence, $P\{\mathrm{SIR}_k^{(N)} \geq \gamma\}$ does not depend on k. Without loss of generality, we study user 1.

We define admissibility for a class of users in the random signature case as follows: when the processing gain is N, α users per degree of freedom of the processing gain are *admissible* in the system if

$$P\left\{\mathrm{SIR}_1^{(N)} \geq \gamma\right\} > a.$$

Similar to the deterministic signature case, the *network capacity* $\alpha_N(a)$ is defined as the maximum number of users (per degree of freedom of the processing gain) that are admissible in the system, that is,

$$\alpha_N(a) \stackrel{\triangle}{=} \sup\left\{\alpha \;\middle|\; P\left\{\mathrm{SIR}_1^{(N)} \geq \gamma\right\} > a\right\}.$$

The *asymptotic network capacity* $\alpha_\infty(a)$ is defined as the limit of $\{\alpha_N(a)\}$

$$\alpha_\infty(a) \stackrel{\triangle}{=} \lim_{N\to\infty} \alpha_N(a).$$

We have the following result on the network capacity for the random signature case.

Theorem 3.2 (Random Signature Case): The asymptotic network capacity $\alpha_\infty(a)$ of a system with MF receivers is

$$\alpha_\infty(a) = \frac{F^{-1}(1-a)}{\gamma\mu} - \frac{\eta}{\mu}.$$

We note that in the special case of perfect power control, the asymptotic network capacity is simply $\frac{1}{\gamma} - \frac{\eta}{\mu}$, which agrees with [27].

We have the following heuristic interpretation of Theorems 3.1 and 3.2. Suppose the received power distribution is F in a large system. Roughly speaking, by optimally allocating signatures, at most $N\left(1+\frac{F^{-1}(1-a)}{\gamma\mu} - \frac{\eta}{\mu}\right)$ users can be admitted into the system without sacrificing their QoS requirements; if the users choose their signatures randomly and independently, then at most $N\left(\frac{F^{-1}(1-a)}{\gamma\mu} - \frac{\eta}{\mu}\right)$ users can be admitted into the system without sacrificing their QoS requirements. We conclude that in the imperfect power control case, the network capacity of a system with optimally allocated signatures is precisely one user per degree of freedom greater than that of a system with random signatures, which indicates that the MF receiver is sensitive to the choice of signature sets. This observation is consistent with that of [31], which is for the perfect power control case.

Alternatively, we have a second approach to study the admissibility and network capacity, which is different in principle from the approach above (our first approach). More specifically, we first define asymptotic admissibility as follows. For the deterministic signature case, we say that α users per degree of freedom of the processing gain are *asymptotically admissible* if

$$\liminf_{N\to\infty} \sup_{\mathcal{X}_N} \min_k P\left\{\mathrm{SIR}_k^{(N)} \geq \gamma\right\} > a.$$

For the random signature case, we say that α users per degree of freedom of the processing gain are *asymptotically admissible* if

$$\liminf_{N\to\infty} P\left\{\mathrm{SIR}_1^{(N)} \geq \gamma\right\} > a.$$

For each case, the asymptotic network capacity is defined as the maximum number of asymptotically admissible users. *A priori*, it is not clear if the two different approaches lead to the same expression for the asymptotic network capacity. It turns out that for both deterministic and random signature cases, the asymptotic network capacities obtained by the two approaches are indeed the same.

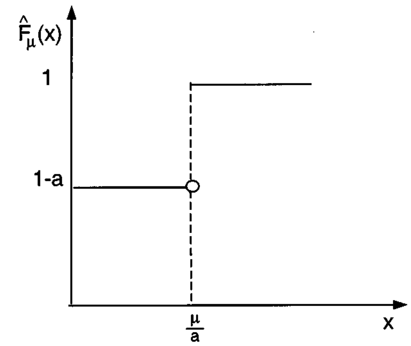

Fig. 2. The distribution that achieves the supremum of $F_\mu^{-1}(1-a)/\mu$.

B. Power Control and Effective Target SIR

Based on Theorems 3.1 and 3.2, we have that for a given received power distribution F, in the deterministic signature case, α users per degree of freedom is asymptotically admissible if $\alpha \leq \frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu}$; in the random signature case, α users per degree of freedom is asymptotically admissible if $\alpha \leq \frac{F^{-1}(1-a)}{\gamma\mu} - \frac{\eta}{\mu}$.

Assuming that a received power distribution of any desired form can be achieved by power control, we now ask a more fundamental question: how many users (per degree of freedom) can be made asymptotically admissible through power control for given SIR requirements? Specifically, we are interested in finding the largest value of $\alpha_\infty(a)$ over all possible received power distributions. Loosely speaking, the above problem can be regarded as the dual of the near–far resistance problem for MF receivers in the following sense: on one hand, near–far resistance is concerned with the worst case performance [29]; on the other hand, optimum power-controlled capacity is concerned with the best case performance.

First we study the deterministic signature case. Define

$$\tilde{\alpha}_d \stackrel{\triangle}{=} \sup_{F(\cdot)} \alpha_\infty(a).$$

Based on Theorem 3.1, we have that

$$\begin{aligned}\tilde{\alpha}_d &= \sup_{F(\cdot)} \left(\frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu}\right)\\ &= \sup_{\mu>0}\left(\sup_{F_\mu(\cdot)}\left(\frac{F_\mu^{-1}(1-a)}{\gamma\mu}\right) + 1 - \frac{\eta}{\mu}\right)\end{aligned} \tag{9}$$

where $F_\mu(\cdot)$ is a distribution with mean μ. Then the calculation of $\tilde{\alpha}_d$ boils down to finding the supremum of the ratio of $F_\mu^{-1}(1-a)$ to μ over all possible distributions with mean μ. It is straightforward to see (e.g., using Markov's Inequality [2, p. 283]) that

$$\sup_{F_\mu(\cdot)} \frac{F_\mu^{-1}(1-a)}{\mu} = \frac{1}{a} \tag{10}$$

where the supremum is achieved by the distribution (denoted as $\hat{F}_\mu$) as shown in Fig. 2. Note that the D^+ corresponding to $\hat{F}_\mu$ is simply the set $\{\mu/a\}$, which is connected.

Combining (10) with (9), we conclude that $\tilde{\alpha}_d = \frac{1}{a\gamma} + 1$. Because $\eta > 0$, the asymptotic network capacity $\alpha_\infty(a)$ for any distribution is strictly less than $\tilde{\alpha}_d$. Therefore, in the deterministic signature case, there exists a distribution function for which α users (per degree of freedom) are admissible if and only if $\alpha < \tilde{\alpha}_d$.

Observe that the optimal power control is in the form of "bang-bang" control. We have the following heuristic interpretation. Suppose the number of users (per degree of freedom) is less than (and close to) $\tilde{\alpha}_d$. Because each user's QoS requirement is that $P\{\text{SIR}^{(N)} \geq \gamma\} > a$, we can implement power control such that the SIR of each user is greater than (and close to) γ with probability a and equals 0 with probability $1-a$. Thus very little power is wasted, i.e., the power control is efficient. Accordingly, when $a < 1$, perfect power control is no longer the best in this context. The reason lies in the fact that we have loosened the users' QoS requirements. When a approaches 1, the optimum power control strategy becomes perfect power control, and $\tilde{\alpha}_d = \frac{1}{\gamma} + 1$, which agrees with the expression given in [31].

For the random signature case, it can be shown that

$$\tilde{\alpha}_r = \sup_{F(\cdot)} \alpha_\infty(a) = \frac{1}{a\gamma}.$$

Thus in the random signature case, there exists a distribution function for which α users (per degree of freedom) are admissible if and only if $\alpha < \tilde{\alpha}_r$. When a approaches 1, the optimum power control strategy becomes perfect power control, and $\tilde{\alpha}_r = \frac{1}{\gamma}$, which is as given in [27].

In a fixed wireless environment, the received power distributions are more or less of the same type (with different parameters) when we scale up/down the transmitted powers; for example, lognormal distributions due to obstacles in the signal paths [33]. Suppose α users per degree of freedom are not admissible for a given received power distribution. Then an interesting question to ask is whether or not it is possible to make α admissible by scaling up the received power and when is it possible. The answer is yes and the underlying intuition is that it is possible to make the users admissible at the expense of more power. To be more specific, we first define a *scale family* (cf., [3, p. 118]).

Definition 2: Let $F(x)$ be any distribution function. Then for any $c > 0$, the family $\mathcal{G}$ of distribution functions $F(x/c)$, indexed by the parameter c, is called the *scale family with standard distribution function* $F(x)$.

Note that for any $G_1, G_2 \in \mathcal{G}, G_1$ is a "scaled version" of G_2, and *vice versa*. A very special (degenerate) example of the scale family is

$$\mathcal{G} = \left\{F_t = I_{[t,\infty)} : t > 0\right\}$$

which essentially represents the totality of distribution functions under perfect power control (here, we use the notation I_A to denote the indicator function of the set A).

Given a scale family $\mathcal{G}$, we define $\gamma' = \gamma c_e$, where $c_e = \frac{\mu_G}{G^{-1}(1-a)}$, $G \in \mathcal{G}$. An important observation is that c_e is fixed for the scale family $\mathcal{G}$. That is, c_e is invariant over all the distributions in $\mathcal{G}$. It follows that γ' is also invariant over all the distributions in $\mathcal{G}$. Therefore, γ' is a property of the whole scale family. We call γ' the *effective target SIR* for $\mathcal{G}$. For example, we have $\gamma' = a\gamma$ for the scale family of distributions that have the form as shown in Fig. 2.

Observe that in a given scale family, there exists a one-to-one correspondence between distribution and mean. Let F_μ and μ denote the corresponding distribution and mean, respectively. We have the following proposition for a given scale family $\mathcal{G}$ with standard distribution function $F(x)$.

Theorem 3.3:

a) (Deterministic signature case) There exists a finite positive value that can be designated as the mean μ such that α users per degree of freedom are asymptotically admissible for F_μ if and only if $\alpha \frac{\gamma'}{1+\gamma'} < 1$. Moreover, the minimum value for μ is

$$\tilde{\mu}_d = \frac{\eta \frac{\gamma'}{1+\gamma'}}{1 - \alpha \frac{\gamma'}{1+\gamma'}}. \tag{11}$$

b) (Random signature case) There exists a finite positive value that can be designated as the mean μ such that α users per degree of freedom are asymptotically admissible for F_μ if and only if $\alpha\gamma' < 1$. Moreover, the minimum value for μ is

$$\tilde{\mu}_r = \frac{\eta\gamma'}{1 - \alpha\gamma'}. \tag{12}$$

The proof of Theorem 3.3 follows directly from Theorems 3.1 and 3.2.

Theorem 3.3 is particularly useful in the scenarios where the users' received powers are random due to imperfect power control. (The SIR requirements are therefore probabilistic.) Specifically, in a given scale family $\mathcal{G}$, α users per degree of freedom are asymptotically admissible in the deterministic (or random) signature case for any distribution with mean no less than $\tilde{\mu}_d$ (or $\tilde{\mu}_r$); conversely, α users per degree of freedom are not asymptotically admissible for any distribution with mean less than $\tilde{\mu}_d$ (or $\tilde{\mu}_r$). Let $\tilde{F}$ denote the distribution function corresponding to $\tilde{\mu}_d$ (or $\tilde{\mu}_r$). An easy observation is that α users are asymptotically admissible when we "scale up" $\tilde{F}$ by any constant greater than 1 (i.e., for any distribution function $\tilde{F}'(x) = \tilde{F}(x/c), c \geq 1$).

Theorem 3.3 also allows us to conclude that even if α users per degree of freedom are not asymptotically admissible for a given distribution F, it is still possible to make α users per degree of freedom asymptotically admissible by scaling up F to the extent that the mean μ exceeds $\tilde{\mu}_d$ (or $\tilde{\mu}_r$). This is because the effective target SIR γ' is invariant for all distributions in $\mathcal{G}$, and the asymptotic network capacity corresponding to a particular distribution F in the deterministic (or random) signature case is simply $1 + \frac{1}{\gamma'} - \frac{\eta}{\mu}$ (or $\frac{1}{\gamma'} - \frac{\eta}{\mu}$). By scaling up F, $\frac{\eta}{\mu}$ decreases, and the asymptotic network capacity approaches the power-unconstrained network capacity $1 + \frac{1}{\gamma'}$ (or $\frac{1}{\gamma'}$).

A more interesting observation is that the power-unconstrained network capacity of an imperfect power-controlled system is of the same form as that of a perfect power-controlled system, except with the target SIR γ replaced by the effective target SIR γ'. Roughly speaking, the effective target SIR γ' plays the same role in determining the network capacity of an imperfect power-controlled system as that of the target SIR γ in determining the network capacity of a perfect power-controlled system. Using the effective target SIR, it is also easy to take into account power constraints that arise naturally in a physical system. We will elaborate further on this point at the end of Section IV.

C. Proofs of Theorems 3.1 and 3.2

The proofs of Theorems 3.1 and 3.2 make use of Egoroff's Theorem [22, p. 73] repeatedly. For ease of reference, we restate Egoroff's Theorem here.

Theorem 3.4: If $\{g_n\}$ is a sequence of measurable functions that converge to a real-valued function g almost surely on a measurable set B of finite measure, then given any $\delta > 0$, there is a subset $A \subset B$ with $mA < \delta$ such that g_n converges to g uniformly on $B\backslash A$. (The symbol mA denotes the Lebesgue measure of A.)

1) Proof of Theorem 3.1, MF Case: Throughout this proof, the signatures are deterministic. As in [31], we want to find the maximum number of users supportable by the system by optimally allocating signature sets. Therefore, in this subsection, we focus on the case $K \geq N$ $(\alpha \geq 1)$, because when $K \leq N$, trivially, orthogonal signature sets suppress multiple-access interference. We begin with some technical lemmas.

Define

$$\mathcal{D}_1(\mathcal{X}_N, k) \triangleq \sum_{i \neq k} \left[P_i^{(N)} - \mu^{(N)}\right] \left(s_k^t s_i\right)^2$$

$$\mathcal{D}_2(\mathcal{X}_N, k) \triangleq \sum_{i \neq k} \mu^{(N)} \left(s_k^t s_i\right)^2.$$

For any sequence of signature sets $\{\mathcal{X}_N\}$, if

$$\limsup_{N\to\infty} \max_k \sum_{i=1}^{K} (s_k^t s_i)^2$$

is not finite, it is straightforward to see that at least one user cannot have its SIR requirements met. Therefore, without loss of generality, we assume that

$$\limsup_{N\to\infty} \max_k \sum_{i=1}^{K} (s_k^t s_i)^2$$

is finite.

Lemma 3.1:

a) For any sequence $\{\mathcal{X}_N\}$ in $\mathcal{F}, \mathcal{D}_1(\mathcal{X}_N, k) \to 0$ almost surely.

b) If $\alpha \geq 1$, then $\max_k \mathcal{D}_2(\mathcal{X}_N, k) \geq (\alpha - 1)\mu^{(N)}$, and equality is achieved when $\mathcal{X}_N$ is a WBE signature set.

Proof: See Appendix A. □

Let $\overline{F^{(N)}}(x) = 1 - F^{(N)}(x)$, $\bar{F}(x) = 1 - F(x)$, and $x_0 = \inf\{x \,|\, x \in D^+\}$. Define

$$\beta_N(a) \triangleq \begin{cases} x_0 & \overline{F^{(N)}}(x_0) \leq a \leq \overline{F^{(N)}}(\eta\gamma) \\ \sup\{x \,|\, \overline{F^{(N)}}(x) > a\} & a < \overline{F^{(N)}}(x_0). \end{cases}$$

We have the following lemma.

Lemma 3.2: If $a \leq 1 - F(\eta\gamma)$, then

$$\lim_{N\to\infty} \beta_N(a) = F^{-1}(1-a).$$

Proof: See Appendix A. □

We now complete the proof of Theorem 3.1.

First we show that $\alpha_\infty(a)$ is upper-bounded by $\frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu}$. By Lemma 3.1, we have that $\mathcal{D}_1(\mathcal{X}_N, k)$ converges to 0 almost surely. Fix $\delta > 0$. Appealing to Egoroff's Theorem, there exists a measurable set A such that $P(A) < \delta$ and $\mathcal{D}_1(\mathcal{X}_N, k)$ converges to 0 uniformly on $\bar{A} = \Omega\backslash A$. Then it follows that for fixed $\epsilon > 0$, there exists an integer $N_0(\epsilon)$ such that for all $N \geq N_0(\epsilon)$, $|\mathcal{D}_1(\mathcal{X}_N, k)| \leq \epsilon$ for every point in $\bar{A}$.

Based on (2), we have that

$$\left\{\text{SIR}_k^{(N)} \geq \gamma\right\} = \left\{P_k^{(N)} \geq \gamma[\eta + \mathcal{D}_2(\mathcal{X}_N, k) + \mathcal{D}_1(\mathcal{X}_N, k)]\right\}.$$

Then for all $N \geq N_0(\epsilon)$

$$\begin{aligned} &P\left\{\text{SIR}_k^{(N)} \geq \gamma\right\} \\ &\quad = P\left\{\left(\text{SIR}_k^{(N)} \geq \gamma\right) \cap A\right\} + P\left\{\left(\text{SIR}_k^{(N)} \geq \gamma\right) \cap \bar{A}\right\} \\ &\quad \leq \delta + P\left\{\left(P_k^{(N)} \geq \gamma[\eta + \mathcal{D}_2(\mathcal{X}_N, k) \right.\right. \\ &\qquad\qquad \left.\left. + \mathcal{D}_1(\mathcal{X}_N, k)]\right) \cap \bar{A}\right\} \\ &\quad \leq \delta + \overline{F^{(N)}}(\gamma(\eta + \mathcal{D}_2(\mathcal{X}_N, k) - \epsilon)). \end{aligned} \tag{13}$$

Combining (13) with the definition of network capacity yields that

$$\max_k \mathcal{D}_2(\mathcal{X}_N, k) \leq \frac{\beta_N(a-\delta)}{\gamma} - \eta + \epsilon.$$

By Lemma 3.1, we have that

$$\alpha_N(a) \leq \frac{1}{\mu^{(N)}} \left(\frac{\beta_N(a-\delta)}{\gamma} - \eta + \epsilon\right) + 1. \tag{14}$$

Appealing to Lemma 3.2, it follows that

$$\lim_{N\to\infty} \beta_N(a-\delta) = F^{-1}(1-a+\delta).$$

Because both ϵ and δ are arbitrary positive numbers, and $\lim_{N\to\infty} \mu^{(N)} = \mu$, we conclude that

$$\alpha_\infty(a) = \lim_{N\to\infty} \alpha_N(a) \leq \frac{F^{-1}(1-a)}{\gamma\mu} - \frac{\eta}{\mu} + 1.$$

Next we show that $\alpha_\infty(a) \geq \alpha$ if there exists a sequence of WBE signature sets in $\mathcal{F}$ corresponding to this α, where

$$\alpha < \frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu}.$$

Let $\alpha = \frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu} - \epsilon_1$, where $\epsilon_1 > 0$. Observe that

$$\lim_{N\to\infty} \gamma\left(\eta + (\alpha-1)\mu^{(N)} + \epsilon\right) = F^{-1}(1-a) + \gamma(\epsilon - \epsilon_1\mu).$$

Moreover, based on Lemma 3.2, we have that

$$\lim_{N\to\infty} \beta_N(a+\delta) = F^{-1}(1-a-\delta).$$

Because F is continuous on D^+, it follows that there exist δ, ϵ, and N_1 such that for all $N \geq N_1$

$$\gamma\left[(\alpha-1)\mu^{(N)} + \eta + \epsilon\right] < \beta_N(a+\delta)$$

which implies that

$$\overline{F^{(N)}}\left(\gamma\left[(\alpha-1)\mu^{(N)}+\eta+\epsilon\right]\right) > a+\delta.$$

Under the choice of a WBE signature set $\mathcal{X}_N$ corresponding to this α, we have that for $k=1,\cdots,K$

$$\mathcal{D}_2(\mathcal{X}_N,k)=(\alpha-1)\mu^{(N)}.$$

Thus it follows that for all $N\geq \max(N_1, N_0(\epsilon))$

$$\begin{aligned} &P\left\{\text{SIR}_k^{(N)}\geq\gamma\right\}\\ &\quad=P\left\{\left(\text{SIR}_k^{(N)}\geq\gamma\right)\cap A\right\}+P\left\{\left(\text{SIR}_k^{(N)}\geq\gamma\right)\cap\bar{A}\right\}\\ &\quad\geq P\left\{\left(P_k^{(N)}\geq\gamma[\eta+\mathcal{D}_2(\mathcal{X}_N,k)+\mathcal{D}_1(\mathcal{X}_N,k)]\right)\cap\bar{A}\right\}\\ &\quad\geq P\left\{\left(P_k^{(N)}\geq\gamma[\eta+\mathcal{D}_2(\mathcal{X}_N,k)+\epsilon]\right)\cap\bar{A}\right\}\\ &\quad\geq\overline{F^{(N)}}\left(\gamma\left(\eta+(\alpha-1)\mu^{(N)}+\epsilon\right)\right)+P(\bar{A})-1\\ &\quad\geq\overline{F^{(N)}}\left(\gamma\left(\eta+(\alpha-1)\mu^{(N)}+\epsilon\right)\right)-\delta\\ &\quad> a \end{aligned}\tag{15}$$

that is,

$$P\{\text{SIR}_k^{(N)}\geq\gamma\}>a,\qquad k=1,\cdots,K.$$

Hence, $\alpha_N(a)\geq\alpha$, which dictates that $\alpha_\infty(a)\geq\alpha$.

We conclude that if for any $\alpha<\frac{F^{-1}(1-a)}{\gamma\mu}+1-\frac{\eta}{\mu}$, there exists a sequence of WBE signature sets in $\mathcal{F}$ with $K/N=\alpha$, then $\alpha_\infty(a)=\frac{F^{-1}(1-a)}{\gamma\mu}+1-\frac{\eta}{\mu}$.

Note that the quantity $\alpha_N(a)$ is rational by definition (though $\alpha_\infty(a)$ is not necessarily rational). To get across the main ideas, we neglected this constraint in the proof. However, because rationals are dense on the real line, our approach leads to the same result.

2) Proof of Theorem 3.1, MMSE Case: Throughout this proof, the signatures are deterministic as well. In what follows, by matrix inequality $\boldsymbol{A}>\boldsymbol{B}$ ($\boldsymbol{A}\geq\boldsymbol{B}$), we mean that $\boldsymbol{A}-\boldsymbol{B}$ is positive definite (semidefinite).

Lemma 3.3: Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are $N\times N$ symmetric matrices, and $\boldsymbol{A}\geq\boldsymbol{B}$. Then for any $x,z\in\mathbb{R}^N$

$$(x^t\boldsymbol{A}z)^2\geq(x^t\boldsymbol{B}z)^2.$$

Proof: See Appendix A. □

Recall that $S=[s_1,\cdots,s_K]$. We have the following lemma on the eigenvalues of SS^t.

Lemma 3.4: Let $\{\nu_1,\cdots,\nu_N\}$ be the eigenvalues of SS^t. Then it follows that

$$\sum_{i=1}^{N}\frac{\nu_i}{\nu_i+c_1}\leq\frac{KN}{K+Nc_1}$$

where c_1 is any positive constant.

The proof of Lemma 3.4 follows essentially the same line as that of [31, Theorem 4.1].

We are now ready for the proof of Theorem 3.1 for the linear MMSE case.

Let $\mathcal{I}_k^{(N)}=s_k^tM_{I,k}^{-1}M'_{I,k}M_{I,k}^{-1}s_k$. First we show that

$$\mathcal{I}_k^{(N)}-s_k^tM_{I,k}^{-1}s_k\overset{\text{a.s.}}{\to}0.\tag{16}$$

To this end, we note that

$$\begin{aligned}\mathcal{I}_k^{(N)}&=s_k^tM_{I,k}^{-1}\left(\sum_{i\neq k}\mu^{(N)}s_is_i^t+\eta I\right)M_{I,k}^{-1}s_k\\ &\quad+s_k^tM_{I,k}^{-1}\left(\sum_{i\neq k}\left[P_i^{(N)}-\mu^{(N)}\right]s_is_i^t\right)M_{I,k}^{-1}s_k\\ &=s_k^tM_{I,k}^{-1}s_k+\sum_{i\neq k}\left[P_i^{(N)}-\mu^{(N)}\right]\left(s_k^tM_{I,k}^{-1}s_i\right)^2.\end{aligned}\tag{17}$$

Because $M_{I,k}^{-1}\leq\frac{1}{\eta}I$, appealing to Lemma 3.3 yields that

$$\left(s_k^tM_{I,k}^{-1}s_i\right)^2\leq\frac{\left(s_k^ts_i\right)^2}{\eta^2}.$$

Combining the above with (3), we conclude that

$$\sup_{\substack{i,k\leq K\\ i\neq k}}\left(s_k^tM_{I,k}^{-1}s_i\right)^2\log N\to 0.$$

Using the same argument as in the proof of part a) of Lemma 3.1, it can be shown that

$$\sum_{i\neq k}\left[P_i^{(N)}-\mu^{(N)}\right]\left(s_k^tM_{I,k}^{-1}s_i\right)^2\overset{\text{a.s.}}{\to}0.\tag{18}$$

Then the desired result (16) follows by combining (17) with (18).

Next we verify that $\alpha_{\infty,\text{ms}}(a)$ is upper-bounded by $\frac{F^{-1}(1-a)}{\gamma\mu}+1-\frac{\eta}{\mu}$. Based on (16), we appeal to Egoroff's Theorem to obtain that for any fixed $\delta>0$, there exists a measurable set A such that $P(A)<\delta$ and $\mathcal{I}_k^{(N)}-s_k^tM_{I,k}^{-1}s_k$ converges to 0 uniformly on $\bar{A}=\Omega\backslash A$. Then it follows that for fixed $\epsilon>0$, there exists an integer $N_0(\epsilon)$ such that for all $N\geq N_0(\epsilon)$, and for every point in $\bar{A}$

$$\left|\mathcal{I}_k^{(N)}-s_k^tM_{I,k}^{-1}s_k\right|\leq\epsilon.$$

In what follows we first establish the following two inequalities:

$$\min_k\frac{1}{s_k^tM_{I,k}^{-1}s_k}\leq\frac{\beta_N(a-\delta)}{\gamma}-\epsilon\tag{19}$$

$$\min_k\frac{1}{s_k^tM_{I,k}^{-1}s_k}\geq(\alpha_N(a)-1)\mu^{(N)}+\eta.\tag{20}$$

Observe that for all $N\geq N_0(\epsilon)$ and for every point in $\bar{A}$,

$$P_k^{(N)}\left(s_k^tM_{I,k}^{-1}s_k-\epsilon\right)\leq\text{SIR}_{k,\text{ms}}^{(N)}\leq P_k^{(N)}\left(s_k^tM_{I,k}^{-1}s_k+\epsilon\right).$$

Then it follows that

$$\begin{aligned}&P\left\{\text{SIR}_{k,\text{ms}}^{(N)}\geq\gamma\right\}\\ &\quad=P\left\{\left(\text{SIR}_{k,\text{ms}}^{(N)}\geq\gamma\right)\cap A\right\}+P\left\{\left(\text{SIR}_{k,\text{ms}}^{(N)}\geq\gamma\right)\cap\bar{A}\right\}\\ &\quad\leq\delta+P\left\{\left(P_k^{(N)}\left(s_k^tM_{I,k}^{-1}s_k+\epsilon\right)\geq\gamma\right)\cap\bar{A}\right\}\\ &\quad\leq\delta+P\left\{P_k^{(N)}\left(s_k^tM_{I,k}^{-1}s_k+\epsilon\right)\geq\gamma\right\}.\end{aligned}$$

Combining the above with the definition of network capacity, we have that for $k = 1, \cdots, K$

$$a - \delta < P\left\{P_k^{(N)}(s_k^t M_{I,k}^{-1} s_k + \epsilon) \geq \gamma\right\}$$

which yields that

$$a - \delta < \overline{F^{(N)}}\left(\frac{\gamma}{s_k^t M_{I,k}^{-1} s_k + \epsilon}\right).$$

Then it is straightforward to see that the inequality (19) holds.

To show the second inequality (20), using the matrix inverse lemma [4, Lemma 12.2], we have that

$$s_k^t M_{I,k}^{-1} s_k = \frac{1}{\mu^{(N)}}\left[-1 + \frac{1}{1 - s_k^t\left(SS^t + \frac{\eta}{\mu^{(N)}} I\right)^{-1} s_k}\right]. \tag{21}$$

Observe that

$$\begin{aligned}
&\sum_{k=1}^{K} s_k^t \left(SS^t + \frac{\eta}{\mu^{(N)}} I\right)^{-1} s_k \\
&\quad = \sum_{k=1}^{K} \text{trace}\left(s_k s_k^t \left(SS^t + \frac{\eta}{\mu^{(N)}} I\right)^{-1}\right) \\
&\quad = \text{trace}\left(SS^t \left(SS^t + \frac{\eta}{\mu^{(N)}} I\right)^{-1}\right) \\
&\quad = \sum_{i=1}^{N} \frac{\nu_i}{\nu_i + \frac{\eta}{\mu^{(N)}}}.
\end{aligned}$$

Appealing to Lemma 3.4, we have that

$$\min_k s_k^t \left(SS^t + \frac{\eta}{\mu^{(N)}} I\right)^{-1} s_k \leq \frac{1}{\alpha + \frac{\eta}{\mu^{(N)}}}. \tag{22}$$

Since the above inequality holds for any $\alpha < \alpha_N(a)$, it is easy to obtain the inequality (20) by combining (22) with (21).

Based on the inequalities (19) and (20), we have that

$$(\alpha_N(a) - 1)\mu^{(N)} + \eta \leq \frac{\beta_N(a - \delta)}{\gamma} - \epsilon$$

which yields that

$$\alpha_N(a) \leq \frac{1}{\mu^{(N)}}\left(\frac{\beta_N(a - \delta)}{\gamma} - \eta - \epsilon\right) + 1.$$

Again, because both ϵ and δ are arbitrary positive numbers, we conclude that

$$\alpha_{\infty,\text{ms}}(a) = \lim_{N\to\infty} \alpha_{N,\text{ms}}(a) \leq \frac{F^{-1}(1 - a)}{\gamma\mu} - \frac{\eta}{\mu} + 1.$$

It remains to show that $\alpha_{\infty,\text{ms}}(a) \geq \alpha$ if there exist a sequence of WBE signature sets in $\mathcal{F}$ corresponding to this α, where $\alpha < \frac{F^{-1}(1-a)}{\gamma\mu} + 1 - \frac{\eta}{\mu}$. It is easy to show that under the choice of a WBE signature set, for $k = 1, \cdots, K$

$$s_k^t M_{I,k}^{-1} s_k = \frac{1}{(\alpha - 1)\mu^{(N)} + \eta}.$$

Then for for all $N \geq N_0(\epsilon)$, we have that

$$\begin{aligned}
&P\left\{\text{SIR}_{k,\text{ms}}^{(N)} \geq \gamma\right\} \\
&\quad = P\left\{\left(\text{SIR}_{k,\text{ms}}^{(N)} \geq \gamma\right) \cap A\right\} + P\left\{\left(\text{SIR}_{k,\text{ms}}^{(N)} \geq \gamma\right) \cap \bar{A}\right\} \\
&\quad \geq P\left\{\left(P_k^{(N)}\left(s_k^t M_{I,k}^{-1} s_k - \epsilon\right) \geq \gamma\right)\right\} - \delta \\
&\quad = \overline{F^{(N)}}\left(\frac{\gamma}{\frac{1}{(\alpha-1)\mu^{(N)}+\eta} - \epsilon}\right) - \delta.
\end{aligned}$$

Using the same argument as in the proof for the MF receiver case, it can be shown that if

$$\alpha < \frac{F^{-1}(1 - a)}{\gamma\mu} + 1 - \frac{\eta}{\mu}$$

then

$$P\left\{\text{SIR}_{k,\text{ms}}^{(N)} \geq \gamma\right\} > a, \qquad k = 1, \cdots, K$$

which implies that $\alpha_N(a) \geq \alpha$. Thus we have that $\alpha_{\infty,\text{ms}}(a) \geq \alpha$, completing the proof.

3) Proof of Theorem 3.2: Throughout the proof of Theorem 3.2, the signatures are random. First we define

$$\mathcal{B}_1^{(N)} \triangleq \frac{1}{K}\sum_{i=2}^{K}[P_i^{(N)} - \mu^{(N)}]\xi_i^2$$

and

$$\mathcal{B}_2^{(N)} \triangleq \frac{1}{K}\sum_{i=2}^{K}\mu^{(N)}\xi_i^2$$

where

$$\xi_i = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} s_{1n}s_{in}, \qquad i = 2, \cdots, K.$$

Based on (2), we have that

$$\text{SIR}_1^{(N)} = \frac{P_1}{\eta + \alpha\left(\mathcal{B}_1^{(N)} + \mathcal{B}_2^{(N)}\right)}. \tag{23}$$

We have the following lemma.

Lemma 3.5: When $N \to \infty$, $\mathcal{B}_1^{(N)}$ converges to 0 and $\mathcal{B}_2^{(N)}$ converges to μ almost surely.

Proof: See Appendix A. □

In what follows, we complete the proof of Theorem 3.2. Because the proof makes use of similar techniques to those in the proof of Theorem 3.1, we omit some details here.

Appealing to Lemma 3.5, we have that $\lim_{N\to\infty}\mathcal{B}_1^{(N)} = 0$ and $\lim_{N\to\infty}\mathcal{B}_2^{(N)} = \mu$ with probability one. Fix $\delta > 0$. By Egoroff's Theorem, there exists a measurable set A_1 such that $P(A_1) < \frac{\delta}{2}$ and $|\mathcal{B}_1^{(N)}|$ converges to 0 uniformly on $\bar{A}_1 = \Omega\backslash A_1$, and there exists a measurable set A_2 such that $P(A_2) < (\delta/2)$ and $\mathcal{B}_2^{(N)}$ converges to μ uniformly on $\bar{A}_2 = \Omega\backslash A_2$.

Let $A = A_1 \cup A_2$ and $\bar{A} = \Omega\backslash A$. Then for fixed $\epsilon > 0$, there exists an integer $N_0(\epsilon)$ such that for all $N \geq N_0(\epsilon)$, the quantity $|\mathcal{B}_1^{(N)}|$ is less than or equal to ϵ for every point in $\bar{A}$;

there exists an integer $N_1(\epsilon)$ such that for all $N \geq N_1(\epsilon)$, and every point in $\bar{A}$

$$\mu - \frac{\epsilon}{\alpha} \leq \mathcal{B}_2^{(N)} \leq \mu + \frac{\epsilon}{\alpha}.$$

Let

$$N_2(\epsilon) = \max(N_0(\epsilon), N_1(\epsilon)).$$

Then we have all $N \geq N_2(\epsilon)$

$$\begin{aligned} &P\left\{\text{SIR}_k^{(N)} \geq \gamma\right\} \\ &\quad = P\left\{\left(\text{SIR}_k^{(N)} \geq \gamma\right) \cap A\right\} + P\left\{\left(\text{SIR}_k^{(N)} \geq \gamma\right) \cap \bar{A}\right\} \\ &\quad \leq \delta + P\left\{\left(P_k^{(N)} \geq \gamma\left[\eta + \mathcal{B}_2^{(N)} + \mathcal{B}_1^{(N)}\right]\right) \cap \bar{A}\right\} \\ &\quad \leq \delta + \overline{F^{(N)}}\left(\gamma\left[\eta + \alpha\mu^{(N)} - 2\epsilon\right]\right) \end{aligned} \tag{24}$$

and

$$\begin{aligned} &P\left\{\text{SIR}_k^{(N)} \geq \gamma\right\} \\ &\quad \geq P\left\{\left(P_k^{(N)} \geq \gamma\left[\eta + \mathcal{B}_2^{(N)} + \mathcal{B}_1^{(N)}\right]\right) \cap \bar{A}\right\} \\ &\quad \geq \overline{F^{(N)}}\left(\gamma\left[\eta + \alpha\mu^{(N)} + 2\epsilon\right]\right) + P(\bar{A}) - 1 \\ &\quad \geq \overline{F^{(N)}}\left(\gamma\left[\eta + \alpha\mu^{(N)} + 2\epsilon\right]\right) - \delta. \end{aligned} \tag{25}$$

Combining (24) with the definition of network capacity

$$\alpha_N(a) \leq \frac{1}{\mu^{(N)}}\left(\frac{\beta_N(a-\delta)}{\gamma} - \eta + 2\epsilon\right).$$

It then follows that

$$\alpha_\infty(a) \leq \frac{F^{-1}(1-a)}{\gamma\mu} - \frac{\eta}{\mu}.$$

Similarly, based on (25), we have that for $N \geq N_2(\epsilon)$

$$\alpha_N(a) \geq \frac{\beta_N(a+\delta)}{\gamma\mu} - \frac{\eta}{\mu} - \frac{2\epsilon}{\mu}.$$

Then we have that

$$\alpha_\infty(a) \geq \frac{F^{-1}(1-a)}{\gamma\mu} - \frac{\eta}{\mu}$$

completing the proof.

IV. Multiple-Class Systems

We have studied the admissibility and network capacity for single-class systems in the previous section. However, future wireless systems will have to support multimedia services such as voice, data, video, and fax. Therefore, it is essential to have a level of generality dealing with users having different QoS requirements.

In a wireless system, the received power distributions depend on time-varying fading and power control algorithms. In particular, short-term variation of the signal power may be modeled by Rician fading when there is a direct line of sight between transmitter and receiver, and by Rayleigh fading when such a path does not exist (see, e.g., [12], [29]). Thus the users close to the base station may have Rician fading while the users away from the base station may have Rayleigh fading. Furthermore, because different users may have different QoS requirements, different power control algorithms may be applied to different users. Therefore, to make our model more general, we assume that different users may have different received power distributions.

Suppose users can be classified into L classes according to their received power distributions (users in each class have the same received power distributions). We use $F_l^{(N)}$ to denote the received power distribution of class l when the processing gain is N, $l = 1, \cdots, L$. We assume that as we scale up the system, $F_l^{(N)}$ converges pointwise to F_l, which denotes the received power distribution of class l in the limiting regime. We let $\mu_l^{(N)}$ and μ_l denote the expectations corresponding to $F_l^{(N)}$ and F_l, respectively. Moreover, we assume that the $F_l^{(N)}$'s and F_l's are continuous on D^+.

Let E_l denote the set of users in class l, and K_l the cardinality of E_l, $l = 1, \cdots, L$. Define $\alpha_l \triangleq K_l/N$, where α_l is taken to be fixed when $N \to \infty$, as in the single-class case. Let $\vec{\alpha} = (\alpha_1, \cdots, \alpha_L)^t$. For convenience, we call the collection $\{F_1, \cdots, F_L\}$ a *group of received power distributions*, and $\vec{P} = (\mu_1, \cdots, \mu_L)^t$ the *mean power vector*.

The SIR achieved by the MF receiver for the ith user in class l can be expressed as

$$\text{SIR}_{il}^{(N)} = \frac{P_{il}^{(N)}}{\eta + \sum\limits_{\substack{n\neq l\\ j\in E_n}} P_{jn}^{(N)}\left(s_{il}^t s_{jn}\right)^2 + \sum\limits_{\substack{k\in E_l\\ k\neq i}} P_{kl}^{(N)}\left(s_{il}^t s_{kl}\right)^2} \tag{26}$$

where the s_{il}'s and the $P_{il}^{(N)}$'s are the signatures and received powers, respectively. The probabilistic model for the users' SIR requirements are

$$P\left\{\text{SIR}_{il}^{(N)} \geq \gamma_l\right\} > a_l, \qquad i \in E_l$$

where $a_l \in [0, 1)$, $l = 1, \cdots, L$. We assume that $a_l \leq 1 - F_l(\eta\gamma_l)$, $l = 1, \cdots, L$.

A. The Deterministic Signature Case

Similar to the single-class case, we assume that every chosen sequence of signature sets satisfies (3), and for $l = 1, \cdots, L$

$$\limsup_{N\to\infty} \max_{i\in E_l} \sum_{n=1}^{L} \sum_{j\in E_n} \left(s_{il}^t s_{jn}\right)^2 < \infty.$$

Then along the same lines as in the proof of Lemma 3.1, it can be shown that

$$\sum_{\substack{n\neq l\\ j\in E_n}} \left[P_{jn}^{(N)} - \mu_n^{(N)}\right]\left(s_{il}^t s_{jn}\right)^2 + \sum_{\substack{k\in E_l\\ k\neq i}} \left[P_{kl}^{(N)} - \mu_l^{(N)}\right]\left(s_{il}^t s_{kl}\right)^2 \overset{\text{a.s.}}{\to} 0. \tag{27}$$

It is desirable to design the signature sets such that all users' QoS requirements are satisfied. When $\sum_{l=1}^L \alpha_l \leq 1$, orthogonal signature sets null out completely multiple-access interference, and each single user transmits data as if it were in a single-user channel. When $\sum_{l=1}^L \alpha_l > 1$, however, a simple closed-form solution to the global optimization of the signature sets in this case seems unattainable. We study a sufficient condition for the admissibility of multiple-class systems in the following.

1) Asymptotic Admissibility: For convenience, first we let $\mathcal{X}_{N,l}$ denote the signature set of class l when the processing gain is N. Given a group of received power distributions, we say a tuple $(\alpha_1, \cdots, \alpha_L)$ is *asymptotically admissible* if

$$\liminf_{N\to\infty} \sup_{\cup_l \mathcal{X}_{N,l}} \min_{i\in E_l} P\{\text{SIR}_{il}^{(N)} \geq \gamma_l\} > a_l, \qquad l = 1, \cdots, L.$$

Following [31], we "channelize" the system and have the following suboptimal scheme: a given processing gain N is divided into L parts and $\sum_{l=1}^{L} N_l = N$, where N_l is the degree of freedom assigned to class l. Moreover, $\lambda_l \triangleq (N_l/N)$ is taken to be fixed as $N \to \infty, l = 1, \cdots, L$. Note that users in different classes do not interfere with one another under this scheme.

Let $\mathcal{I}_{il} = s_{il}^t \boldsymbol{D} s_{jn}$, where

$$\boldsymbol{D} = \sum_{n=1}^{L} \sum_{j\in E_n} s_{jn} s_{jn}^t \mu_n^{(N)}.$$

Observing that for $i \in E_l, l = 1, \cdots, L$

$$\text{SIR}_{il}^{(N)} = \frac{P_l^{(N)}}{\eta + \mathcal{I}_{il} - \mu_l^{(N)}}$$

we are motivated to minimize the maximum among $\{\mathcal{I}_{il}, i \in E_l, l = 1, \cdots, L\}$ over possible partitions of the processing gain and choices of signature sets. Intuitively, we want to suppress the interference as much as possible for all the users simultaneously.

Given a partition of the processing gain, we have that when $(K_l/N_l) \geq 1$

$$\max_{i\in E_l} \mathcal{I}_{il} \geq \frac{\alpha_l}{\lambda_l} \mu_l^{(N)}$$

and equality holds if and only if the signature set for class l is a WBE signature set. Moreover, if the signature set for class l is a WBE signature set, the $\mathcal{I}_{il}$'s are the same for all the users in class l.

In what follows, we first solve the following optimization problem:

$$\min_{\{\lambda_1,\cdots,\lambda_L\}} \max_l \frac{\alpha_l}{\lambda_l} \mu_l \tag{28a}$$

$$\text{subject to} \quad \begin{cases} \sum_{l=1}^{L} \lambda_l = 1 \\ \lambda_l > 0, \qquad l = 1, \cdots, L. \end{cases} \tag{28b}$$

Let $b_l = \alpha_l \mu_l$ and $\lambda = 1/\sum_{l=1}^{L} b_l$. By appealing to [6, Theorem 2.1, p. 114], we have the following lemma, which is used in Theorem 4.1.

Lemma 4.1: Let $\lambda_l^* = b_l \lambda$, $l = 1, \cdots, L$. Then the vector $(\lambda_1^*, \cdots, \lambda_L^*)$ is a minimax point for (28a) under the constraint (28b). Moreover, we have that

$$\frac{\alpha_l}{\lambda_l^*} \mu_l = \sum_{j=1}^{L} \alpha_j \mu_j, \qquad l = 1, \cdots, L.$$

Proof: See Appendix A. □

In a practical spread-spectrum system, it is necessary to have $N_l = \lfloor \lambda_l N \rfloor$. In the asymptotic setting, however, the impact of the difference between N_l and $\lfloor N_l \rfloor$ on the solution to the minimax problem disappears.

Now we are ready to give the following result on the asymptotic admissibility of multiple classes in the deterministic signature case.

Theorem 4.1 (Deterministic Signature Case): A tuple $(\alpha_1, \cdots, \alpha_L)$ is asymptotically admissible if it satisfies

$$\max_l \mu_l \leq \sum_{j=1}^{L} \alpha_j \mu_j \tag{29a}$$

$$\sum_{j=1}^{L} \alpha_j \mu_j \leq \min_l \left(\frac{F_l^{-1}(1 - a_l)}{\gamma_l} + \mu_l \right) - \eta. \tag{29b}$$

Theorem 4.1 provides a sufficient condition for the asymptotic admissibility of multiple-class systems with MF receivers. It is clear that the above condition is also a sufficient condition for the asymptotic admissibility of multiple-class systems with linear MMSE receivers because the SIR achieved by the MF receiver is always no greater than that achieved by the linear MMSE receiver.

Proof: Given a processing gain N, we partition it into L parts such that $N_l^* = \lambda_l^* L, l = 1, \cdots, L$, where λ_l^* is as defined in Lemma 4.1. Observe that for $l = 1, \cdots, L$

$$\mu_l \leq \sum_{j=1}^{L} \alpha_j \mu_j \Leftrightarrow \frac{b_l}{\sum_{l=1}^{L} b_l} \leq \alpha_l \Leftrightarrow \frac{N_l^*}{K_l} \leq 1. \tag{30}$$

Therefore, under the above partitioning of the processing gain, we can choose a WBE signature set for each class, and users in different classes do not interfere with one another.

Let P_l denote a random variable that has distribution $F_l, l = 1, \cdots, L$. Because $F_l^{(N)}$ converges pointwise to F_l, we have that $P_{il}^{(N)}$ converges in distribution to P_l, $l = 1, \cdots, L$ (see [2, p. 192]). Then under the choice of a WBE signature set for each class, we have that by Lemma 4.1, for $i \in E_l$ and $l = 1, \cdots, L$

$$\mathcal{I}_{il} = \frac{\alpha_l}{\lambda_l^*} \mu_l^{(N)} = \sum_{l=1}^{L} \alpha_l \mu_l^{(N)}. \tag{31}$$

Combining (26), (27), and (31), we conclude that

$$\text{SIR}_{il}^{(N)} \overset{\mathcal{D}}{\Rightarrow} \frac{P_l}{\eta + \sum_{j=1}^{L} \alpha_j \mu_j - \mu_l}.$$

Based on the definition of asymptotic admissibility for $(\alpha_1, \cdots, \alpha_N)$, it suffices to show that under our proposed scheme of designing signature sets

$$\lim_{N\to\infty} P\left\{\text{SIR}_{1l}^{(N)} \geq \gamma_l\right\} > a_l, \qquad l = 1, \cdots, L.$$

In the following, we assume that D^+ is an interval. (When D^+ is a single point, the problem boils down to the admissibility under perfect power control, which has been solved in [31].) First suppose D^+ is an open interval. Then appealing to [1, Theorem 2.1] yields that

$$\lim_{N\to\infty} P\left\{\text{SIR}_{1l}^{(N)} \ge \gamma_l\right\} = P\left\{\lim_{N\to\infty} \text{SIR}_{1l}^{(N)} \ge \gamma_l\right\}.$$

It follows that for $l = 1, \cdots, L$

$$\begin{aligned}&\lim_{N\to\infty} P\left\{\text{SIR}_{1l}^{(N)} \ge \gamma_l\right\} > a_l\\ &\Leftrightarrow P\left\{P_l \ge \gamma_l\left(\eta + \sum_{j=1}^{L} \alpha_j\mu_j - \mu_l\right)\right\} > a_l\\ &\Leftrightarrow \sum_{j=1}^{L} \alpha_j\mu_j \le \frac{F_l^{-1}(1-a_l)}{\gamma_l} + \mu_l - \eta\end{aligned}$$

which is equivalent to

$$\sum_{j=1}^{L} \alpha_j\mu_j \le \min_l \left(\frac{F_l^{-1}(1-a_l)}{\gamma_l} + \mu_l\right) - \eta. \tag{32}$$

Combining the above approach with Egoroff's Theorem, it can be shown that (32) holds when D^+ is a closed (or half-closed) interval. This completes the proof. □

2) Power Control, Effective Target SIR, and Effective Bandwidth: As in the single-class case, it is possible to have the users' SIR requirements satisfied by scaling up the received powers even if the users are not admissible for a given group of distributions. For each class, we introduce the scale family (with standard distribution 3 function F_l)

$$\mathcal{G}_l = \{F_l(p_l/c), c > 0\}, \qquad l = 1, \cdots, L.$$

Define $\gamma_l' = \gamma_l c_{el}$, where

$$c_{el} = (\mu_{G_l}/G_l^{-1}(1-a_l)), \qquad G_l \in \mathcal{G}_l.$$

We note that γ_l' is invariant over all the distributions in $\mathcal{G}_l$ and is a property of $\mathcal{G}_l$. We call γ_l' the *effective target SIR* for $\mathcal{G}_l$. For convenience, we call the collection $\{\mathcal{G}_1, \cdots, \mathcal{G}_L\}$ a *group of scale families*.

Using the notation of effective target SIR, (29b) can be written as

$$\mu_l \ge \gamma_l'\left[\eta + \sum_{j=1}^{L} \alpha_j\mu_j - \mu_l\right], \qquad l = 1, \cdots, L.$$

Define

$$\begin{aligned}\vec{\Gamma} &\triangleq (\gamma_1', \cdots, \gamma_L')^t\\ \vec{B} &\triangleq \left(\frac{1}{1+\gamma_1'}, \cdots, \frac{1}{1+\gamma_L'}\right)^t\\ \vec{P}_d &\triangleq ((1+\gamma_1')\mu_1, \cdots, (1+\gamma_L')\mu_L)^t.\end{aligned}$$

Using the above notation, (29b) further boils down to

$$\vec{P}_d \ge (\vec{\alpha} \odot \vec{B})^t \vec{P}_d \vec{\Gamma} + \eta\vec{\Gamma} \tag{33}$$

where the symbol $\odot$ denotes the operator for Hadamard product, which simply performs the element-wise multiplication of two matrices [24]. Note that inequality of vectors is equivalent to component-wise inequalities.

We want to study the feasibility of (33), that is, the condition under which there exist positive vectors $\vec{P}_d$ satisfying (33). Based on [17], [19], [36], we draw the conclusion that a necessary and sufficient condition for the existence of a finite positive solution to (33) is the existence of a finite positive solution to the system of equations obtained by setting all inequalities in (33) to equalities. This remarkable result is due to [25, Theorem 2.1] and a strong result called "The Subinvariance Theorem" [25, Theorem 1.6]. More specifically, we have the following lemma.

Lemma 4.2: There exists a finite positive vector $\vec{P}_d$ satisfying (33) if and only if

$$\sum_{l=1}^{L} \alpha_l \frac{\gamma_l'}{1+\gamma_l'} < 1. \tag{34}$$

Moreover, the lth component of the component-wise minimum mean power vector satisfying (33) is

$$\mu_l = \frac{\eta\frac{\gamma_l'}{1+\gamma_l'}}{1 - \sum_{l=1}^{L} \alpha_l \frac{\gamma_l'}{1+\gamma_l}}, \qquad l = 1, \cdots, L. \tag{35}$$

Proof: Observe that $\vec{\Gamma}(\vec{\alpha} \odot \vec{B})^t$ is a nonnegative irreducible matrix and has rank one. Thus based on [25, Theorem 1.1], the Perron–Frobenius eigenvalue of $\vec{\Gamma}(\vec{\alpha} \odot \vec{B})^t$ is $(\vec{\alpha} \odot \vec{B})^t\vec{\Gamma}$. Appealing to [17, Proposition 2.1], we have that a necessary and sufficient condition for the existence of a nonnegative nonzero solution to (33) is

$$\sum_{l=1}^{L} \frac{\gamma_l'}{1+\gamma_l'}\alpha_l < 1.$$

Define

$$\vec{P}_d^* \triangleq (I - \vec{\Gamma}(\vec{\alpha} \odot \vec{B})^t)^{-1}\eta\vec{\Gamma}.$$

Then the solution $\vec{P}_d^*$ is Pareto optimal in the following sense: any other feasible solution to (33) will have every component not less and at least one component greater than the solution $\vec{P}_d^*$.

It is straightforward to see that

$$\begin{aligned}\vec{P}_d^* &= (I - \vec{\Gamma}(\vec{\alpha} \odot \vec{B})^t)^{-1}\eta\vec{\Gamma}\\ &= \frac{\eta}{1 - (\vec{\alpha} \odot \vec{B})^t\vec{\Gamma}}\vec{\Gamma}.\end{aligned} \tag{36}$$

Recalling the relationship between $\vec{P}_d$ and the mean power vector, we have that the lth component of the component-wise minimum mean power vector is

$$\mu_l = \frac{\eta\frac{\gamma_l'}{1+\gamma_l'}}{1 - \sum_{l=1}^{L} \alpha_l \frac{\gamma_l'}{1+\gamma_l}}$$

completing the proof. □

Observe that in a given group of scale families, there exists a one-to-one correspondence between group of received power distributions and mean power vector. Given a group of scale families $\{\mathcal{G}_1, \cdots, \mathcal{G}_L\}$, we let $\{F_{\mu_1}, \cdots, F_{\mu_L}\}$ and $\{\mu_1, \cdots, \mu_L\}$ denote the corresponding group of distributions and mean power vector, respectively. Based on Theorem 4.1 and Lemma 4.2, we have the following result for a given group of scale families $\{\mathcal{G}_1, \cdots, \mathcal{G}_L\}$.

Proposition 4.1: There exists a finite positive vector that can be assigned as the mean power vector $\{\mu_1, \cdots, \mu_L\}$ such that a tuple $(\alpha_1, \cdots, \alpha_L)$ is asymptotically admissible for $\{F_{\mu_1}, \cdots, F_{\mu_L}\}$ if

$$\max_l \frac{\gamma_l'}{1+\gamma_l'} \leq \sum_{l=1}^{L} \alpha_l \frac{\gamma_l'}{1+\gamma_l'}$$

and

$$\sum_{l=1}^{L} \alpha_l \frac{\gamma_l'}{1+\gamma_l'} < 1.$$

Proof: Substituting (35) into (29a), we have that for $l = 1, \cdots, L$

$$\mu_l \leq \sum_{l=1}^{L} \alpha_l \mu_l \Leftrightarrow \frac{\eta \frac{\gamma_l'}{1+\gamma_l'}}{1 - \sum_{l=1}^{L} \alpha_l \frac{\gamma_l'}{1+\gamma_l}} \leq \frac{\eta \sum_{l=1}^{L} \alpha_l \frac{\gamma_l'}{1+\gamma_l}}{1 - \sum_{l=1}^{L} \alpha_l \frac{\gamma_l'}{1+\gamma_l}}$$
$$\Leftrightarrow \frac{\gamma_l'}{1+\gamma_l'} \leq \sum_{l=1}^{L} \alpha_l \frac{\gamma_l'}{1+\gamma_l'}.$$

If the two conditions specified in Proposition 4.1 are satisfied, then by applying Theorem 4.1 and Lemma 4.2, the tuple $(\alpha_1, \cdots, \alpha_L)$ is asymptotically admissible if the power vector with the lth component given in (35) is designated as the mean power vector. □

Observing the conditions given in Proposition 4.1, we follow [27] and define the *effective bandwidth* of class l in the deterministic signature case as $\mathcal{E}(\gamma_l) = \frac{\gamma_l'}{1+\gamma_l'}$ degrees of freedom per user. However, we are able to give only a sufficient condition for asymptotic admissibility of a tuple $(\alpha_1, \cdots, \alpha_L)$ in terms of effective bandwidth. Further work is needed for this case.

It is easy to see that the effective bandwidth is a monotonically increasing function of the effective target SIR. Under the optimum power control stated in the single-class case, $\gamma_l' = a_l \gamma_l, l = 1, \cdots, L$. It follows that the conditions given in Proposition 4.1 become the following:

$$\max_l \frac{a_l\gamma_l}{1+a_l\gamma_l} \leq \sum_{l=1}^{L} \alpha_l \frac{a_l\gamma_l}{1+a_l\gamma_l},$$

and

$$\sum_{l=1}^{L} \alpha_l \frac{a_l\gamma_l}{1+a_l\gamma_l} < 1.$$

B. The Random Signature Case

As in our study of single-class systems with random signatures, we apply Lemma A.1 to conclude that

$$\text{SIR}_{il}^{(N)} \overset{\mathcal{D}}{\Rightarrow} \frac{P_l}{\eta + \sum_{l=1}^{L} \alpha_l \mu_l}, \qquad \text{as } N \to \infty. \tag{37}$$

Without loss of generality, we study the first user in class l, $l = 1, \cdots, L$.

1) Asymptotic Admissibility: Given a group of received power distributions, we say a tuple $(\alpha_1, \cdots, \alpha_L)$ is *asymptotically admissible* if

$$\lim_{N\to\infty} P\left\{\text{SIR}_{1l}^{(N)} \geq \gamma_l\right\} > a_l, \qquad l = 1, \cdots, L.$$

(That the above limit exists is shown in the proof of Theorem 4.2 below.)

We have the following result on the asymptotic admissibility of multiple classes in the random signature case.

Theorem 4.2 (Random Signature Case): A tuple $(\alpha_1, \cdots, \alpha_L)$ is asymptotically admissible if and only if

$$\sum_{l=1}^{L} \alpha_l \mu_l \leq \min_l \left(\frac{F_l^{-1}(1-a_l)}{\gamma_l} \right) - \eta. \tag{38}$$

Proof: First suppose D^+ is an open interval. Appealing to [1, Theorem 2.1], we have that

$$\lim_{N\to\infty} P\left\{\text{SIR}_{1l}^{(N)} \geq \gamma_l\right\} = P\left\{\lim_{N\to\infty} \text{SIR}_{1l}^{(N)} \geq \gamma_l\right\}.$$

By definition, a tuple $(\alpha_1, \cdots, \alpha_N)$ is asymptotically admissible if and only if for $l = 1, \cdots, L$

$$\lim_{N\to\infty} P\left\{\text{SIR}_{1n}^{(N)} \geq \gamma_l\right\} > a_l.$$

Thus we have that for $l = 1, \cdots, L$

$$\lim_{N\to\infty} P\left\{\text{SIR}_{1l}^{(N)} \geq \gamma_l\right\} > a_l$$
$$\Leftrightarrow P\left\{P_l \geq \gamma_l \left(\eta + \sum_{l=1}^{L} \alpha_l \mu_l\right)\right\} > a_l$$
$$\Leftrightarrow \sum_{l=1}^{L} \alpha_l \mu_l \leq \frac{F_l^{-1}(1-a_l)}{\gamma_l} - \eta$$

which is equivalent to

$$\sum_{l=1}^{L} \alpha_l \mu_l \leq \min_l \left(\frac{F_l^{-1}(1-a_l)}{\gamma_l} \right) - \eta. \tag{39}$$

Combining the above approach with Egoroff's Theorem, it can be shown that (39) holds when D^+ is a closed (or half-closed) interval. The proof is completed. □

Alternatively, Theorem 4.2 may also be proved by applying [27, Proposition 3.3], but the proof would be more involved. We note that different from the deterministic signature case (in which we provide only a sufficient condition on the admissibility of multiple-class systems), we can also follow our first approach developed in the single-class case to define the *network capacity region* in the random signature case as

$$\mathcal{C}_N(a_1, \cdots, a_L) \triangleq \left\{(\alpha_1, \cdots, \alpha_L) \,\middle|\, P\left\{\text{SIR}_{1l}^{(N)} \geq \gamma_l\right\} > a_l, \; l = 1, \cdots, L\right\}.$$

The *asymptotic network capacity region* is defined as

$$\mathcal{C}_\infty(a_1, \cdots, a_L) \triangleq \lim_{N\to\infty} \mathcal{C}_N(a_1, \cdots, a_L).$$

(It turns out that the above limit exists.)

Following the same argument as in the single-class case, it can be shown that the asymptotic network capacity region is the following simplex:

$$\left\{(\alpha_1, \cdots, \alpha_L) \,\middle|\, \sum_{l=1}^{L} \alpha_l \mu_l \le \min_l \left(\frac{F_l^{-1}(1-a_l)}{\gamma_l}\right) - \eta \right\}$$

which is consistent with Theorem 4.2. We omit the details here.

2) Power Control, Effective Target SIR, and Effective Bandwidth: Using matrix notation, the inequality (38) can be simplified to

$$\vec{P} \ge \vec{\alpha}^t \vec{P}\vec{\Gamma} + \eta\vec{\Gamma}. \tag{40}$$

We now state the following lemma regarding the feasibility of (40) without proof because it follows the same line of proof as that of Lemma 4.2.

Lemma 4.3: There exists a finite positive vector $\vec{P}$ satisfying (40) if and only if

$$\sum_{l=1}^{L} \alpha_l \gamma_l' < 1. \tag{41}$$

Moreover, the lth component of the component-wise minimum mean power vector satisfying (40) is

$$\mu_l = \frac{\eta\gamma_l'}{1 - \sum_{l=1}^{L} \alpha_l \gamma_l'}, \qquad l = 1, \cdots, L. \tag{42}$$

Based on Lemma 4.3, we can easily obtain the following result for a given group of scale families $\{\mathcal{G}_1, \cdots, \mathcal{G}_L\}$.

Proposition 4.2: There exists a finite positive vector that can be assigned as the mean power vector $\{\mu_1, \cdots, \mu_L\}$ such that a tuple $(\alpha_1, \cdots, \alpha_L)$ is asymptotically admissible for $\{F_{\mu_1}, \cdots, F_{\mu_L}\}$ if and only if

$$\sum_{l=1}^{L} \alpha_l \gamma_l' < 1.$$

Moreover, the lth component of the component-wise minimum mean power vector is

$$\mu_l = \frac{\eta\gamma_l'}{1 - \sum_{l=1}^{L} \alpha_l \gamma_l'}, \qquad l = 1, \cdots, L.$$

The proof of Proposition 4.2 follows by combining the results of Lemma 4.3 and Theorem 4.2.

Proposition 4.2 is a generalization of a result given in [27]. Similar to the deterministic signature case, we define the *effective bandwidth* of class l for the random signature case as $\mathcal{E}(\gamma_l) = \gamma_l'$ degrees of freedom per user. Under the optimum power control stated in the single-class case, the condition given in (41) boils down to $\sum_{l=1}^{L} \alpha_l a_l \gamma_l < 1$.

We note that in both the deterministic and random signature cases, the effective bandwidth is a monotonically increasing function of the effective target SIR. As observed in the single-class case, the effective target SIR plays the same role in determining the network capacity of an imperfect power-controlled system as that of the target SIR in determining the network capacity of a perfect power-controlled system. Hence, we conclude that the effective bandwidth in terms of the effective target SIR plays the same role in determining the admissibility of an imperfect power-controlled system as that of the effective bandwidth in terms of the target SIR in determining the admissibility of a perfect power-controlled system. More specifically, Proposition 4.2 tells us that for a given group of scale families, a tuple $(\alpha_1, \cdots, \alpha_N)$ can be made asymptotically admissible through power control if and only if the sum of effective bandwidth of all classes is less than one.

In a practical wireless system, the transmitted powers and hence received powers naturally are constrained. Suppose the users in class l have an average power constraint that $\mu_l \le \bar{\mu}_l$, $l = 1, \cdots, L$. Given a group of scale families, it is straightforward to see that there exists a finite positive vector (satisfying the power constraints) that can be assigned as the mean power vector $\{\mu_1, \cdots, \mu_L\}$ such that $(\alpha_1, \cdots, \alpha_L)$ is asymptotically admissible for $\{F_{\mu_1}, \cdots, F_{\mu_L}\}$ if and only if

$$\sum_{l=1}^{L} \alpha_l \gamma_l' \le \min_l \left(1 - \frac{\eta\gamma_l'}{\bar{\mu}_l}\right).$$

V. Conclusion

We have studied single-cell symbol-synchronous CDMA systems in fading channels, focusing primarily on the scenarios where MF receivers are employed. We adopt a probabilistic model for the users' QoS requirements. For single-class systems with MF receivers, we identify the network capacities for both deterministic and random signature cases. The network capacity can be expressed uniquely in terms of the users' SIR requirements and the received power distribution. For multiple-class systems with MF receivers, we provide a necessary and sufficient condition for the random signature case, but only a sufficient condition for the deterministic signature case, for a set of users per degree of freedom to be admissible.

We have explored the notions of *effective target SIR* γ' and *effective bandwidth* $\mathcal{E}(\gamma')$. In particular, given a scale family $\mathcal{G}$, γ' is invariant over all the distributions in $\mathcal{G}$, and plays the same role in determining the admissibility of an imperfect power-controlled system as that of the target SIR γ in determining the admissibility of a perfect power-controlled system. The effective bandwidth $\mathcal{E}(\gamma')$, a monotonically increasing function of γ', is particularly useful when there are multiple classes of users in an imperfect power-controlled system.

We have also characterized the network capacity of single-class systems with linear MMSE receivers for the deterministic signature case. A similar characterization for the random signature case turns out to be highly nontrivial; we are looking into this problem currently.

Our results are useful for network-level resource-allocation problems such as admission control and power control in a large network. It is easy to generalize these results to obtain the network capacity of CDMA systems that employ the techniques of sectorization and voice-activity monitoring (see [7]).

In this paper, we have focused on characterizing the network capacity so that we can determine how many users can be accommodated without sacrificing their QoS requirements. On the other hand, another fundamental issue is the *channel capacity*. That is, how much information can be transmitted reliably through fading channels in CDMA systems equipped with linear receivers? Some work along this line is already underway (e.g., [30]). Our own preliminary study shows that there exists a tradeoff between network capacity and channel capacity. Intuitively, the more users in the systems, the stronger the MAI that is imposed by other users, and hence less information that can be transmitted reliably over the channel.

It must also be pointed out that our results are for single-cell synchronous systems, as is the case in [27], [31]. Further work is needed to extend these results to multiple-cell asynchronous systems.

APPENDIX A
PROOFS OF TECHNICAL LEMMAS

We use the following strong law for triangular arrays repeatedly, which follows directly from [5, Lemma 2.4]:

Lemma A.1: Let $X_{in}, i = 1, \cdots, n;\ n = 1, 2, \cdots$ be a triangular array of random variables defined on a common probability space such that for each n, $\{X_{in}, i = 1, \cdots, n\}$ are independent, and let $S_n = \sum_{i=1}^n X_{in}$. If $S_n/n \to 0$ in probability and

$$\sum_{i=1}^{n} \frac{X_{in}^2 \log n}{n^2} \to 0 \quad \text{a.s.}$$

then $S_n/n \to 0$ almost surely.

A. Proof of Lemma 3.1

Proof for part a): Let $\rho_{ki}^2 = (s_k^t s_i)^2$. Then

$$\mathcal{D}_1(\mathcal{X}_N, k) = \sum_{i \neq k} [P_i^{(N)} - \mu^{(N)}] \rho_{ki}^2.$$

By Lemma A.1, it suffices to show that $\mathcal{D}_1(\mathcal{X}_N, k)$ converges to 0 in probability and

$$\sum_{i \neq k} \left[P_i^{(N)} - \mu^{(N)}\right]^2 \left(\rho_{ki}^2\right)^2 \log N \overset{\text{a.s.}}{\to} 0.$$

(Note that $K = \alpha N$ and α is fixed.)

By assumption, $|P_i^{(N)} - \mu^{(N)}|$ is bounded by d with probability one, which implies that $\mathrm{var}\,(P_i^{(N)})$ is bounded by d^2. Because

$$\lim \sup_{N \to \infty} \max_k \sum_{j=1}^{K} \rho_{ki}^2 < \infty$$

we have that

$$\begin{aligned} E\left[(\mathcal{D}_1(\mathcal{X}_N, k))^2\right] &= \sum_{i \neq k} \left(\rho_{ki}^2\right)^2 \mathrm{var}\Big(P_i^{(N)}\Big) \\ &\le \left(\sup_{\substack{i,k \le K \\ i \neq k}} \rho_{ki}^2\right) d^2 \left(\sum_{i \neq k} \rho_{ki}^2\right) \\ &\to 0 \end{aligned}$$

where the last step follows from the fact that

$$\lim_{N \to \infty} \sup_{\substack{i,k \le K \\ i \neq k}} \rho_{ki}^2 \log N = 0.$$

Then it follows that $\mathcal{D}_1(\mathcal{X}_N, k)$ converges to 0 in probability because mean-square convergence implies convergence in probability.

Observe that

$$\sum_{i \neq k} \left[P_i^{(N)} - \mu^{(N)}\right]^2 \left(\rho_{ki}^2\right)^2 \log N \le \left(\sup_{\substack{i,k \le K \\ i \neq k}} \rho_{ki}^2 \log N\right) d^2 \sum_{\substack{i=1 \\ i \neq k}}^{K} \rho_{ki}^2 \to 0$$

holds with probability one. We conclude that $\mathcal{D}_1(\mathcal{X}_N, k) \to 0$ almost surely.

Proof for part b): By Welch's bound [16], [34], it follows that

$$\sum_{k=1}^{K} \sum_{i=1}^{K} (s_k^t s_i)^2 \ge \frac{K^2}{N} \tag{43}$$

which implies that

$$\sum_{k=1}^{K} \sum_{\substack{i=1 \\ i \neq k}}^{K} \left(s_k^t s_i\right)^2 \ge \frac{K^2}{N} - K.$$

Then we have that

$$\max_k \sum_{\substack{i=1 \\ i \neq k}}^{K} \left(s_k^t s_i\right)^2 \ge \alpha - 1$$

which yields that

$$\max_k \mathcal{D}_2(\mathcal{X}_N, k) \ge (\alpha - 1)\mu^{(N)} \tag{44}$$

where the equality in (44) is achieved when $\mathcal{X}_N$ is a WBE signature set.

B. Proof of Lemma 3.2

To show that

$$\lim_{N \to \infty} \beta_N(a) = F^{-1}(1-a)$$

it suffices to verify that

$$\lim \sup_{N \to \infty} \beta_N(a) \le F^{-1}(1-a)$$

and

$$\lim \inf_{N \to \infty} \beta_N(a) \ge F^{-1}(1-a).$$

First we consider the interior part of D^+. Suppose

$$\lim \inf_{N \to \infty} \beta_N(a) < F^{-1}(1-a).$$

Choose s, t such that

$$\lim \inf_{N \to \infty} \beta_N(a) < s < t < F^{-1}(1-a).$$

Then there exists $N_1 < N_2 < \cdots$ such that

$$\beta_{N_j}(a) < s < t < F^{-1}(1-a)$$

for all $j = 1, 2, \cdots$. Observing that $\overline{F^{(N)}}$ is nonincreasing, we have that

$$\overline{F^{(N_j)}}\left(\beta_{N_j}(a)\right) \geq \overline{F^{(N_j)}}(s) \geq \overline{F^{(N_j)}}(t) \geq \overline{F^{(N_j)}}(F^{-1}(1-a)). \qquad (45)$$

By the definition of $\beta_N(a)$, we have that $\overline{F^{(N_j)}}(\beta_{N_j}(a)) = a$ because $\overline{F^{(N_j)}}$ is continuous on D^+. Moreover, because $F^{(N)}$ converges pointwise to F on D^+, we have that

$$\lim_{j\to\infty} \overline{F^{(N_j)}}(s) = \overline{F}(s)$$
$$\lim_{j\to\infty} \overline{F^{(N_j)}}(t) = \overline{F}(t)$$

and

$$\lim_{j\to\infty} \overline{F^{(N_j)}}(F^{-1}(1-a)) = a.$$

Since F is strictly increasing on D^+, it then follows that $\overline{F}$ is strictly decreasing on D^+. We take limit of (45) to conclude that

$$a \geq \overline{F}(s) > \overline{F}(t) \geq a.$$

Then we have a contradiction, which dictates that

$$\liminf_{N\to\infty} \beta_N(a) \geq F^{-1}(1-a).$$

Using similar arguments, we have that

$$\limsup_{N\to\infty} \beta_N(a) \leq F^{-1}(1-a).$$

Therefore, it follows that

$$\lim_{N\to\infty} \beta_N(a) = F^{-1}(1-a).$$

Along the same lines, it can be shown that the result holds at the endpoints of D^+ when D^+ is closed (or half-closed).

C. Proof of Lemma 3.3

Because $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric and $\boldsymbol{A} \geq \boldsymbol{B}$, it follows that

$$\boldsymbol{A}^2 \geq \boldsymbol{B}^2.$$

Thus for any $z \in \mathbb{R}^N$, it is straightforward to see that

$$(\boldsymbol{A}z)^t \boldsymbol{A}z \geq (\boldsymbol{B}z)^t \boldsymbol{B}z.$$

That is,

$$\|\boldsymbol{A}z\|_2^2 \geq \|\boldsymbol{B}z\|_2^2.$$

Observe that $\boldsymbol{A}zz^t\boldsymbol{A}^t$ and $\boldsymbol{B}zz^t\boldsymbol{B}^t$ are positive semidefinite with eigenvalues $\{\|\boldsymbol{A}z\|_2^2, 0, \cdots, 0\}$ and $\{\|\boldsymbol{B}z\|_2^2, 0, \cdots, 0\}$, respectively. Then it follows that

$$\boldsymbol{A}zz^t\boldsymbol{A}^t \geq \boldsymbol{B}zz^t\boldsymbol{B}^t$$

which implies that for any $x \in \mathbb{R}^N$

$$x^t\boldsymbol{A}zz^t\boldsymbol{A}^t x \geq x^t\boldsymbol{B}zz^t\boldsymbol{B}^t x$$

thereby proving Lemma 3.3.

D. Proof of Lemma 3.5

Note that the signature s_1 is known to the MF receiver for user 1. For a given s_1, the ξ_i's are i.i.d., and

$$E[\xi_i^2 \,|\, s_1] = \frac{1}{N}\sum_n s_{1n}^2, \qquad i = 2, \cdots, K.$$

By the Strong Law of Large Numbers

$$\frac{1}{N}\sum_n s_{1n}^2 \xrightarrow{\text{a.s.}} 1.$$

Moreover, we have that conditioned on s_1

$$\frac{1}{K}\sum_{i=2}^K \xi_i^2 - E\left[\xi_i^2 \,\middle|\, s_1\right] \xrightarrow{\text{a.s.}} 0.$$

Thus it follows that

$$\frac{1}{K}\sum_{i=2}^K \xi_i^2 \xrightarrow{\text{a.s.}} 1$$

which yields that $\mathcal{B}_2^{(N)} \xrightarrow{\text{a.s.}} \mu$.

It what follows, we show $\mathcal{B}_1^{(N)} \to 0$ almost surely by using Lemma A.1. Observe that

$$\begin{aligned}
E\xi_i^4 &= \frac{1}{N^2}\sum_{k_1}\sum_{k_2}\sum_{k_3}\sum_{k_4} E\left[s_{1k_1}s_{1k_2}s_{1k_3}s_{1k_4}s_{ik_1}s_{ik_2}s_{ik_3}s_{ik_4}\right]\\
&\leq \frac{1}{N^2}\Bigg\{\sum_{k_1} E\left[s_{1k_1}^4 s_{ik_1}^4\right]\\
&\quad + \sum_{k_1}\sum_{k_2\neq k_1} E\left[s_{1k_1}^2 s_{1k_2}^2 s_{ik_1}^2 s_{ik_2}^2\right]\\
&\quad + \sum_{k_1}\sum_{k_3\neq k_1} E\left[s_{1k_1}^2 s_{1k_3}^2 s_{ik_1}^2 s_{ik_3}^2\right]\\
&\quad + \sum_{k_1}\sum_{k_4\neq k_1} E\left[s_{1k_1}^2 s_{1k_4}^2 s_{ik_1}^2 s_{ik_4}^2\right]\Bigg\}\\
&= 3 + \frac{\left(E\left[s_{11}^4\right]\right)^2}{N}\\
&< \infty
\end{aligned} \qquad (46)$$

where we have used the assumption that $E[s_{11}^4] < \infty$. Note that s_1 and $\{s_2, \cdots, s_K\}$ are independent. Then it follows that for almost every realization of s_1

$$E\left[\xi_i^4 \,\middle|\, s_1\right] < \infty.$$

(All statements involving conditional expectation hold with probability one by default.)

Since $|P_i^{(N)} - \mu^{(N)}| \leq d$, it follows that for almost every realization of s_1, $\lim_{N\to\infty} E[(\mathcal{B}_1^{(N)})^2 | s_1] = 0$, which implies that $\lim_{N\to\infty} E[(\mathcal{B}_1^{(N)})^2] = 0$. Thus $\mathcal{B}_1^{(N)}$ converges to 0 in probability. Then it remains to show that

$$\frac{\log N}{N^2}\sum_{i=2}^K \left[P_i^{(N)} - \mu^{(N)}\right]^2 \xi_i^4 \xrightarrow{\text{a.s.}} 0.$$

Observe that

$$\frac{\log N}{N^2}\sum_{i=2}^K \left[P_i^{(N)} - \mu^{(N)}\right]^2 \xi_i^4 \leq \frac{d^2 \log N}{N^2}\sum_{i=2}^K \xi_i^4.$$

By the Strong Law of Large Numbers, we have that for almost every realization of s_1

$$\frac{1}{K}\sum_{i=2}^K \xi_i^4 - E\left[\xi_i^4 \,\middle|\, s_1\right] \xrightarrow{\text{a.s.}} 0$$

which implies that

$$\frac{\log N}{N^2}\sum_{i=2}^{K}\left[P_i^{(N)}-\mu^{(N)}\right]^2\xi_i^4 \overset{\text{a.s.}}{\to} 0.$$

Then it follows that for almost every realization of s_1

$$\mathcal{B}_1^{(N)}=\frac{1}{K}\sum_{i=2}^{K}\left[P_i^{(N)}-\mu^{(N)}\right]\xi_i^2 \overset{\text{a.s.}}{\to} 0.$$

Therefore, we have that $\mathcal{B}_1^{(N)} \to 0$ almost surely, completing the proof of Lemma 3.5.

E. Proof of Lemma 4.1

We use the following lemma, which follows directly from [6, Theorem 2.1, p. 114] and [6, Property III, p. 51]:

Lemma A.2: Let $f_l(x), l \in \{1,\cdots,L\}$, be continuously differentiable convex functions on a convex closed set Ω in $\mathbb{R}^n$. Consider the problem

$$\min_{x\in\Omega}\max_{l\in\{1,\cdots,L\}} f_l(x).$$

A point $X^* \in \Omega$ is a minimax point for the above problem (i.e., a solution to the above optimization problem) if and only if

$$\inf_{z\in\Omega}\max_{l\in R(X^*)}\left\langle\frac{\partial f_l(X^*)}{\partial x}, z-X^*\right\rangle = 0 \tag{47}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$, and

$$R(x)=\left\{l\in\{1,\cdots,L\}\;\middle|\; f_l(x)=\max_{l\in\{1,\cdots,L\}} f_l(x)\right\}.$$

We are now ready to prove Lemma 4.1. Fix ϵ such that $0<\epsilon<\min_{l\in\{1,\cdots,L\}}\lambda_l^*$. Define

$$\Omega=\left\{(\lambda_1,\cdots,\lambda_L)\;\middle|\;\sum_{l=1}^{L}\lambda_l=1,\ \lambda_l\ge\epsilon,\ l=1,\cdots,L\right\}.$$

It is easy to verify that Ω is a closed convex set and that $X^*=(\lambda_1^*,\cdots,\lambda_L^*)$ lies in Ω.

Let $x=(\lambda_1,\cdots,\lambda_L)$. Define $f_l(x)\triangleq(b_l/\lambda_l), l=1,\cdots,L$. We note that $f_l(x)$ is convex on Ω. In the following we verify that X^* satisfies the condition specified in Lemma A.2. Let $(z-X^*)=(z_1-\lambda_1^*,\cdots,z_L-\lambda_L^*)$. We have that

$$\frac{\partial f_l(x)}{\partial x}=\left(0,\cdots,0,\frac{-b_l}{\lambda_l^2},0,\cdots,0\right)$$

which yields that

$$\frac{\partial f_l(X^*)}{\partial x}=\left(0,\cdots,0,\frac{-1}{b_l\lambda^2},0,\cdots,0\right).$$

It then follows that

$$\left\langle\frac{\partial f_l(X^*)}{\partial x}, z-X^*\right\rangle=\frac{-1}{b_l\lambda^2}(z_l-\lambda_l^*),\qquad l=1,\cdots,L.$$

Note that the expression on the left-hand side of (47) cannot be positive since we can always take $z=X^*$ and get a zero inner product. We prove by contradiction that the condition in (47) is satisfied by X^*.

Suppose (47) fails to hold for X^*. Then there exists a point $Z=(z_1,\cdots,z_L)\in\Omega$ such that

$$\max_{l\in R(X^*)}\left\langle\frac{\partial f_l(X^*)}{\partial x}, Z-X^*\right\rangle<0.$$

Because $f_l(X^*)=\sum_{l=1}^{L}\alpha_l\mu_l$ for $l=1,\cdots,L$, we have that

$$\max_{l\in R(X^*)}\left\langle\frac{\partial f_l(X^*)}{\partial x}, Z-X^*\right\rangle<0$$
$$\Leftrightarrow\quad\max_{l\in\{1,\cdots,N\}}\left\langle\frac{\partial f_l(X^*)}{\partial x}, Z-X^*\right\rangle<0$$

that is, for $l=1,\cdots,L$

$$\left\langle\frac{\partial f_l(X^*)}{\partial x}, Z-X^*\right\rangle<0.$$

Then it follows that

$$\left\langle\frac{\partial f_l(X^*)}{\partial x}, Z-X^*\right\rangle=\frac{-1}{b_l\lambda^2}(z_l-\lambda_l^*)<0,\qquad l=1,\cdots,L.$$

Since $b_l>0$, we have that $z_l>\lambda_l^*, l=1,\cdots,L$. Thus we have that

$$\sum_{l=1}^{L}z_l>\sum_{l=1}^{L}\lambda_l^*=1$$

which implies that Z is not in Ω, a contradiction. Therefore, X^* satisfies (47).

Because ϵ can be chosen arbitrarily between 0 and $\min_{l\in\{1,\cdots,L\}}\lambda_l^*$, we conclude that X^* is a minimax point for (28a) over the constraint in (28b), and $(\alpha_l/\lambda_l^*)\mu_l=\sum_{l=1}^{L}\alpha_l\mu_l$, $l=1,\cdots,L$.

APPENDIX B
EXAMPLES OF SEQUENCES OF WBE SIGNATURE SETS SATISFYING (3)

In practical spread-spectrum systems, the processing gain N is of the form 2^m. When N is large, the contribution of one component in the signatures to the crosscorrelation is negligible. Thus we consider N of form 2^m-1 without essential loss of generality.

As in [16], we associate a binary sequence $B=[b_1,b_2,\cdots,b_N]$ in GF $(2)^N$ with a signature

$$s=(1/\sqrt{N})[w_1,w_2,\cdots,w_N]$$

where the component-wise mapping is as follows:

$$b_k=\begin{cases}0, & \text{if } w_k=1\\ 1, & \text{if } w_k=-1.\end{cases}$$

Let s and s' be two arbitrary signatures of length $N=2^m-1$, and B and B' the corresponding binary sequences in GF $(2)^N$. Then $(s^ts')=(N-2d(B,B')/N)$, where $d(\cdot,\cdot)$ denotes the Hamming distance between the indicated vectors.

Example 1: Let U denote a maximum-length shift-register code of length $N=2^m-1$ where $m\ge 2$, and $U^\perp$ its dual code. Then U is an (N,m) linear code in which all codewords except the all-zero codeword have identical weight 2^{m-1} [20, p. 435]. Moreover, $U^\perp$ is a Hamming code with minimum distance 3. By [16, Corollary to Proposition 2], a binary signature set corresponding to a binary linear code is a WBE signature set if the minimum distance of its dual code is at least three. Thus the signature set (say $\mathcal{X}$) corresponding to U is a WBE set.

Let s and s' be two arbitrary and different signatures chosen from $\mathcal{X}$. Then $d(B,B') = 2^{m-1}$, which implies that $s^t s' = \frac{N-2d(B,B')}{N} = \frac{-1}{N}$. It follows that as $N \to \infty$

$$\sup\{(s^t s')^2 \log N \mid s, s' \in \mathcal{X}, s \neq s'\} \to 0.$$

In this example, $\lim_{N\to\infty} K/N = 1$.

Example 2: Let W denote a triple-error-correcting primitive narrow-sense binary BCH code of length $N = 2^m - 1$, and $W^\perp$ its dual code. First we construct a linear code U (which is a k-dimensional subspace of $W^\perp$) by using the weight distribution of $W^\perp$ [35, p. 185].

Consider the case $m \geq 5$, m odd (the weight distribution of $W^\perp$ for $N = 2^m - 1$, $m \geq 5$, m odd, is given in [35, Table 8-2(a), p. 185]. Suppose we choose x, y, z, u, and v codewords having weights $2^{m-1} - 2^{(m+1)/2}$, $2^{m-1} - 2^{(m-1)/2}$, 2^{m-1}, $2^{m-1} + 2^{(m-1)/2}$, and $2^{m-1} - 2^{(m+1)/2}$, respectively. Moreover, it is required that $x + y + z + u + w = 2^k - 1$ so that we can construct an (N, k) block code U by using the above $2^k - 1$ nonzero codewords plus all-zero codeword. Using the MacWilliams Identity in [35, p. 90], we can get the weight enumerator of the dual code $U^\perp$. We can choose proper values for x, y, z, u, and v so that the coefficients A_1 and A_2 in the weight enumerator of $U^\perp$ are zero. In general, this can be achieved since we have five variables but only three constraints. (Because the bounds on x, y, z, u, and v given in [35, Table 8-2(a), p. 185] are much larger than N even when m is moderately large, we can choose x, y, z, u, and v as needed.) Therefore, there is no codeword having weight 1 or 2 in the dual code $U^\perp$. By [16, Corollary to Proposition 2], the signature set (say $\mathcal{X}$) corresponding to U is a WBE set.

Along the same line, for the case $m \geq 5$, m even, by exploiting [35, Table 8-2(b), p. 185], we can construct a linear code U to which the corresponding signature set is a WBE set.

Let s and s' be two arbitrary signatures chosen from the WBE set corresponding to U, and B and B' the corresponding binary sequences in GF $(2)^N$. Based on [35, Tables 8-2(a) and 8-2(b) , p. 185], we have that

$$\sup\{|(s^t s')| \mid s, s' \in \mathcal{X}, s \neq s'\} < 4N^{-1/2}$$

which implies that as $N \to \infty$

$$\sup\{(s^t s')^2 \log N \mid s, s' \in \mathcal{X}, s \neq s'\} \to 0.$$

In this example, $\alpha = \lim_{N\to\infty} K/N$ can be chosen to be 2, 4, etc.

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References

[1] P. Billingsley, *Convergence of Probability Measures*. New York: Wiley, 1968.
[2] ——, *Probability and Measure*, 2nd ed. New York: Wiley, 1987.
[3] G. Casella and R. L. Berger, *Statistical Inference*. Belmont, CA: Duxbury, 1990.
[4] E. K. P. Chong and S. H. Zak, *An Introduction to Optimization*. New York: Wiley, 1996.
[5] J. Cuzick, "A strong law for weighted sums of i.i.d. random variables," *J. Theor. Probab.*, vol. 8, no. 3, pp. 625–641, 1995.
[6] V. Dem'yanov and V. Malozemov, *Introduction to Minimax*. New York: Wiley, 1974.
[7] K. S. Gilhousen, I. M. Jacobs, R. Padovani, A. J. Viterbi, L. A. Weaver Jr., and C. E. Wheatley III, "On the capacity of a cellular CDMA system," *IEEE Trans. Veh. Technol.*, vol. 40, pp. 303–312, May 1991.
[8] A. Goldsmith and P. Varaiya, "Capacity of fading channels with channel side information," *IEEE Trans. Inform. Theory*, vol. 43, pp. 1986–1992, Nov. 1997.
[9] S. V. Hanly, "Capacity and power control in spread spectrum macrodiversity radio networks," *IEEE Trans. Commun.*, vol. 44, pp. 247–256, Feb. 1996.
[10] ——, "Information Capacity of Radio Networks," Ph.D. dissertation, Cambridge Univ., Cambridge, U.K., Aug. 1993.
[11] S. V. Hanly and D. Tse, "Multi-access fading channels: Shannon and delay-limited capacities," in *Proc. 33rd Allerton Conf.*, Oct. 1995, pp. 786–795.
[12] M. Hellebrandt and R. Mathar, "Cumulated interference power and bit-error-rates in mobile packet radio," *Wireless Networks*, vol. 3, pp. 169–172, 1997.
[13] R. Lupas and S. Verdú, "Linear multiuser detectors for synchronous code-division multiple access," *IEEE Trans. Inform. Theory*, vol. 35, pp. 123–136, Jan. 1989.
[14] U. Madhow and M. L. Honig, "MMSE interference suppression for directed-sequence spread-spectrum CDMA," *IEEE Trans. Commun.*, vol. 42, pp. 3178–3188, Dec. 1994.
[15] ——, "On the average near-far resistance for MMSE detection of directed-sequence CDMA signals with random spreading," *IEEE Trans. Inform. Theory*, vol. 45, pp. 2039–2045, Sept. 1999.
[16] J. L. Massey and T. Mittelholzer, "Welch's bound and sequence sets for code-division multiple-access systems," in *Sequences II, Methods in Communications, Security and Computer Science*, R. Capocelli, A. D. Santis, and U. Vaccaroo, Eds. New York: Springer-Verlag, 1993.
[17] D. Mitra, "An asynchronous distributed algorithm for power control for cellular radio systems," in *Proc. 4th WINLAB Workshop on Third Generation Wireless Information Networks*, J. M. Holtzman, Ed. Norwell, MA: Kluwer Academic, 1993, pp. 249–257.
[18] D. Mitra and J. A. Morrison, "A distributed power control algorithm for bursty transmissions on cellular, spread spectrum wireless networks," in *Proc. 5th WINLAB Workshop on Third Generation Wireless Information Networks*, J. M. Holtzman, Ed. Norwell, MA: Kluwer Academic, 1996, pp. 201–212.
[19] D. Mitra and J. A. Morrison, "A novel distributed power control algorithm for classes of service in cellular CDMA networks," in *Proc. 6th WINLAB Workshop on Third Generation Wireless Information Networks*, J. M. Holtzman, Ed. Norwell, MA: Kluwer Academic, 1997, pp. 187–202.
[20] J. G. Proakis, *Digital Communications*, 3rd ed. New York: McGraw-Hill, 1995.
[21] Z. Rosberg and J. Zander, "Toward a framework for power control in cellular systems," *Wireless Networks*, vol. 4, no. 3, pp. 215–222, Apr. 1998.
[22] H. L. Royden, *Real Analysis*, 3rd ed. Englewood Cliffs, NJ: Prentice-Hall, 1988.
[23] D. V. Sarwate, "Meeting the Welch bound with equality," in *Sequences and Their Applications: Proc. SETA'98*, C. Ding, T. Helleseth, and H. Niederreiter, Eds. Berlin, Germany: Springer-Verlag, 1999.
[24] J. R. Schott, *Matrix Analysis for Statistics*. New York: Wiley, 1997.
[25] E. Seneta, *Non-Negative Matrices and Markov Chains*, 2nd ed. Berlin, Germany: Springer-Verlag, 1981.
[26] J. W. Silverstein and Z. D. Bai, "On the empirical distribution of eigenvalues of a class of large dimensional random matrices," *J. Multivariate Anal.*, vol. 54, no. 2, pp. 175–192, 1995.
[27] D. Tse and S. V. Hanly, "Linear multiuser receivers: Effective interference, effective bandwidth and user capacity," *IEEE Trans. Inform. Theory*, vol. 45, pp. 641–657, Mar. 1999.

[28] S. Ulukus and R. D. Yates, "Stochastic power control for cellular radio systems," *IEEE Trans. Commun.*, vol. 46, pp. 784–798, June 1998.

[29] S. Verdú, *Multiuser Detection*. Cambridge, U.K.: Cambridge Univ. Press, 1998.

[30] S. Verdú and S. Shamai (Shitz), "Spectral efficiency of CDMA with random spreading," *IEEE Trans. Inform. Theory*, vol. 45, pp. 622–640, Mar. 1999.

[31] P. Viswanath, V. Anantharam, and D. Tse, "Optimal sequences, power control, and user capacity of synchronous CDMA systems with linear MMSE multiuser receivers," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1968–1983, Sept. 1999.

[32] A. J. Viterbi, *CDMA—Principles of Spread Spectrum Communications*. Reading, MA: Addison-Wesley, 1995.

[33] A. J. Viterbi and R. Padovani, "Implications of mobile cellular CDMA," *IEEE Commun. Mag.*, pp. 38–41, Dec. 1992.

[34] L. R. Welch, "Lower bounds on the maximum cross-correlation of signals," *IEEE Trans. Inform. Theory*, vol. IT-20, pp. 397–399, May 1974.

[35] S. B. Wicker, *Error Control Systems for Digital Communications and Storage*. Englewood Cliffs, NJ: Prentice Hall, 1995.

[36] R. D. Yates, "A framework for uplink power control in cellular radio systems," *IEEE J. Select. Areas Commun.*, vol. 13, pp. 1341–1348, Sept. 1995.