When Network Effect Meets Congestion Effect: Leveraging Social Services for Wireless Services

Xiaowen Gong
School of Electrical, Computer and Energy Engeering
Arizona State University
Tempe, AZ 85287, USA
xgong9@asu.edu

Lingjie Duan
Engineering Systems and Design Pillar
Singapore University of Technology and Design
Singapore 487372, Singapore
lingjie_duan@sutd.edu.sg

Xu Chen*
Institute of Computer Science
University of Goettingen
Goettingen 37073, Germany
xu.chen@cs.uni-goettingen.de

ABSTRACT

The recent development of social services tightens wireless users' social relationships and encourages them to generate more data traffic under network effect. This boosts the demand for wireless services yet may challenge the limited wireless capacity. To fully exploit this opportunity, we study mobile users' data usage behaviors by jointly considering the network effect based on their social relationships in the social domain and the congestion effect in the physical wireless domain. Accordingly, we develop a Stackelberg game for problem formulation: In Stage I, a wireless provider first decides the data pricing to all users to maximize its revenue, and then in Stage II users observe the price and decide data usage subject to mutual interactions under both network and congestion effects. We analyze the two-stage game using backward induction. For Stage II, we first show the existence and uniqueness of a user demand equilibrium (UDE). Then we propose a distributed update algorithm for users to reach the UDE. Furthermore, we investigate the impacts of different parameters on the UDE. For Stage I, we develop an optimal pricing algorithm to maximize the wireless provider's revenue. We evaluate the performance of our proposed algorithms by numerical studies using real data, and thereby draw useful engineering insights for the operation of wireless providers.

Categories and Subject Descriptors

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Keywords

Mobile social network, wireless data usage, network effect, congestion effect

1. INTRODUCTION

The past few years have witnessed pervasive penetration of mobile devices in people's daily life, thanks to the wireless technology advances. Motivated by many social applications on mobile platforms (e.g., WeChat, WhatsApp [1, 2]), mobile users' data usage behaviors have been increasingly influenced by their social relationships. In 2014, the number of online social media users on mobile platforms has reached 1.6 billion, accounting for 44% of mobile users and 80% of online social media users [3].

The popularity of social services on mobile platforms also gives opportunities to wireless service providers who operate the mobile networks. Intuitively, social services can encourage mobile users to demand more data usage by stimulating their interactions with each other through these services (e.g., online social gaming and blogging). When a user increases its activity in a social service, its social friends would also increase their activities. Therefore, users' data usage levels for social services present network effect to others [4]. This demand increase provides a great potential for wireless providers' revenue increase.

However, this potential benefit is subject to the limited wireless capacity in physical communication networks (e.g., spectrum). As users increase their data usage, they also experience more congestion (e.g., service delays), which discourages them to use more. The increasing congestion poses a significant challenge for wireless providers to increase their revenues.

As a result, mobile users' data usage behaviors are not only subject to congestion effect in the physical network, but also network effect in the social network (as illustrated in Fig. 1), which has been largely overlooked by traditional wireless providers. To fully exploit the potential benefit brought by social services, it is necessary to investigate users' data usage behaviors in these two domains, so that a wireless provider can take the best strategy in favor of its revenue. With this insight, we not only analyze users' interactions subject to both network and congestion effects,

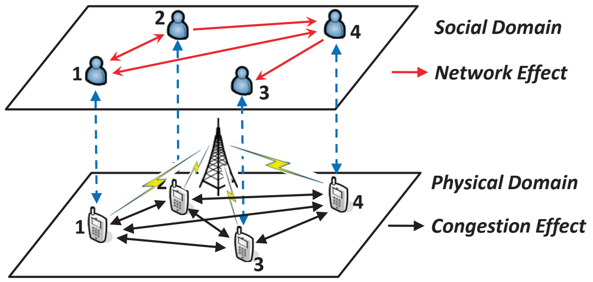


Figure 1: Mobile users experience network effect in the social network and congestion effect in the physical network.

but also study the optimal pricing strategy for the wireless provider.

The main contributions of this paper can be summarized as follows.

- *Stackelberg game formulation:* By jointly considering users' social relationships and the wireless network's congestion, we formulate the interaction between the wireless provider and mobile users as a Stackelberg game: In Stage I, the wireless provider chooses a price to maximize its revenue; in Stage II, mobile users choose their data usage levels based on the price to maximize their socially-aware payoffs.

- *Equilibrium analysis for user demands in Stage II:* We first give a general condition under which there exists a user demand equilibrium (UDE), and then we show that under a further general condition the game is a concave game and thus admits a unique UDE. We also propose a distributed algorithm for users to achieve the UDE. Next we show that a user's usage can increase when price increases. We further show that if the social network is symmetric, the total usage always increases when a user's parameter (e.g., social tie) improves.

- *Provider's optimal pricing in Stage I:* Finally, by taking into account users' equilibrium demands, we develop an optimal pricing algorithm to maximize the revenue of the wireless provider. We evaluate the performance of total usage and revenue by simulations, and draws useful engineering insights for the wireless provider's operation.

The rest of this paper is organized as follows. In Section 2, we formulate the Stackelberg game between the wireless provider and mobile users. Section 3 studies users' demand equilibrium in Stage II. Section 4 studies the provider's optimal pricing strategy in Stage I. Simulation results and discussions are given in Section 5. Related work are reviewed in Section 6. Section 7 concludes this paper.

2. SYSTEM MODEL

2.1 Socially-aware Wireless Service

Consider a set of users $\mathcal{N} \triangleq \{1, \ldots, N\}$ participating in a wireless service provided by a wireless operator (e.g., AT&T). Each user $i \in \mathcal{N}$ consumes an amount of data usage in the wireless service, denoted by x_i where $x_i \in [0, \infty)$. Let $\mathbf{x} \triangleq (x_1, \ldots, x_N)$ denote the usage profile of all the users and $\mathbf{x}_{-i}$ denote the the usage profile without user i. Affected by the other users' usage subject to congestion effect due to the limited resources in the wireless network, the payoff of user i by consuming data usage x_i is

$$v_i(x_i, \mathbf{x}_{-i}, p) = a_i x_i - \frac{1}{2} b_i x_i^2 - \frac{1}{2} c(\sum_{j \in \mathcal{N}} x_j)^2 - p x_i,$$

where $a_i > 0$ and $b_i > 0$ are the *internal utility coefficients* that capture the intrinsic value of the wireless service to user i, $c > 0$ is the *congestion coefficient* that is determined by the resource constraints of the wireless network, and p is the usage-based price charged by the wireless provider[1]. As in [5], the quadratic form of the internal utility function not only allows for tractable analysis, but also serves as a good second-order approximation for a broad class of concave utility functions. In particular, a_i models the maximum internal demand rate, and b_i models the internal demand elasticity factor. For the congestion model, the quadratic sum form reflects that a user's congestion experience is affected by all the users, and the marginal cost of congestion increases as the total usage increases.

Traditional wireless providers' operation does not take into account the fact that social services encourage mobile users to demand more data usage. We thus account for this effect in our model. Then user i's payoff includes the addition of social utility, i.e.,

$$u_i(x_i, \mathbf{x}_{-i}, p) = a_i x_i - \frac{1}{2} b_i x_i^2 + \sum_{j \neq i} g_{ij} x_i x_j - \frac{1}{2} c(\sum_{j \in \mathcal{N}} x_j)^2 - p x_i \quad (1)$$

where $g_{ij} \geq 0$ is the *social tie* that quantifies the social influence from user j to user i. As in [5], the product form $g_{ij} x_i x_j$ of the social utility function captures that a user derives more utility by increasing its usage in social services, and the marginal gain of social utility increases as its social friends increase their usage. Therefore, social services bring in network effect among users and can increase their utilities.

2.2 Stackelberg Game Formulation

We model the interaction between the wireless provider and mobile users for the socially-aware wireless service as a two-stage Stackelberg game.

Definition 1 (Two-Stage Pricing-Usage Game).

- Stage I (Pricing): *The wireless provider chooses price p to maximize its revenue:*

$$p^* = \arg \max_{p \in [0, \infty)} t(\mathbf{x}) p$$

where $t(\mathbf{x}) \triangleq \sum_{i \in \mathcal{N}} x_i$ denotes the total usage under strategy profile $\mathbf{x}$;

- Stage II (Usage): *Each user $i \in \mathcal{N}$ chooses its data usage level x_i to maximize its payoff given the price p and the usage levels of the other users $\mathbf{x}_{-i}$:*

$$x_i^* = \arg \max_{x_i \in [0, \infty)} u_i(x_i, \mathbf{x}_{-i}, p).$$

We study the two-stage pricing-usage game by backward induction [6]. For Stage II, given a price chosen by the wireless provider in Stage I, we are interested in the existence of

[1]Usage-based pricing is widely used in practice by wireless operators to control the demand. Here the price is the same for all users to ensure fairness.

a stable outcome of users' interactions at which no user will deviate. This leads to the concept of user equilibrium.

Definition 2 (User Demand Equilibrium). *For any price p given in Stage I, the user demand equilibrium (UDE) in Stage II is a strategy profile $\mathbf{x}^*$ such that no user can improve its payoff by unilaterally changing its usage, i.e.,*

$$x_i^* = \arg \max_{x_i \in [0,\infty)} u_i(x_i, \mathbf{x}_{-i}^*, p),\ \forall i.$$

Given the UDE in Stage II, we will study the optimal pricing strategy for the wireless provider in Stage I.

3. STAGE II: USER DEMAND EQUILIBRIUM

In this section, we study users' demands in Stage II.

Using the concave payoff function (1), by setting the derivative $\frac{\partial u_i(x_i,\mathbf{x}_{-i})}{\partial x_i} = 0$ as the first-order condition, we obtain the *best response function* of user i as

$$r_i(\mathbf{x}_{-i}) = \max\left\{0,\ \frac{a_i - p}{b_i + c} + \sum_{j \neq i} \frac{g_{ij} - c}{b_i + c} x_j\right\}. \tag{2}$$

According to (2), each user i's usage demand consists of two parts: *internal demand* $\frac{a_i - p}{b_i + c}$ that is independent of the other users, and *external demand* $\sum_{j \neq i} \frac{g_{ij} - c}{b_i + c} x_j$ that depends on the other users. The coefficient $\frac{g_{ij} - c}{b_i + c}$ represents the marginal increase or decrease of user i's demand when user j's usage increases: when $g_{ij} > c$, user j's influence to user i is dominated by network effect; when $g_{ij} < c$, it is dominated by congestion effect.

3.1 Existence and Uniqueness of UDE

We first investigate the existence of UDE in Stage II. We make the following assumption.

Assumption 1. $\sum_{j \neq i} \frac{|g_{ij} - c|}{b_i + c} < 1,\ \forall i.$

This assumption is for analysis tractability. It is an important condition to guarantee the existence of UDE, as there can exist no UDE when it does not hold (as illustrated by an example in Fig. 2). According to the best response function (2), Assumption 1 implies that any user's absolute external demand $|\sum_{j \neq i} \frac{g_{ij} - c}{b_i + c} x_j|$ is less than the maximum usage $\max_{j \neq i} x_j$ among all the other users. This is a mild condition as the aggregate effect experienced by a user from all the other users would be less than the largest effect the user can experience from an individual of the other users. A similar assumption is made in [5] for similar considerations.

Now we can show that there always exists a UDE in Stage II.

Theorem 1. *Under Assumption 1, the Stage II game admits a UDE.*

The proof is given in Appendix and the main idea is to show that the game has an equivalent game which admits a UDE.

Next we give another general technical condition under which the game admits a unique UDE.

Theorem 2. *Under Assumption 1, the Stage II game admits a unique UDE if* $\sum_{j \neq i} \frac{|g_{ji} - c|}{b_i + c} < 1,\ \forall i.$

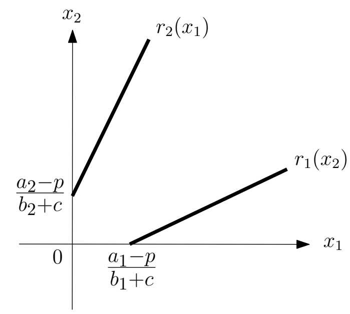


Figure 2: For Stage II for two users, when $\frac{g_{12}-c}{b_1+c} < -1$ and $\frac{g_{21}-c}{b_2+c} < -1$, there does not exist a UDE. The bold lines are the best response functions and do not intersect.

The proof is given in Appendix and the main idea is is to show that the game is a concave game [7], and thus admits a unique UDE.

Remark: According to Theorem 2, it is worth noting that the Stage II game admits a unique UDE when users' social ties are symmetric (i.e., $g_{ij} = g_{ji},\ \forall i \neq j$). The symmetric setting of social networks is of great interests. Motivated by the idea of social reciprocity [8], a user's social behavior to another is likely to imitate the latter's behavior to the former. As a result, two users' social ties to each other tend to be the same.

3.2 Computing and Achieving UDE

As we have showed the existence of UDE, we then design an algorithm to compute the UDE, as described in Algorithm 1. The algorithm iteratively updates users' strategies based on their best response functions (2) and converges to the UDE.

Algorithm 1: Compute the UDE in Stage II

input: precision threshold ϵ;
$x_i^{(0)} \leftarrow 0,\ \forall i \in \mathcal{N}$; $t \leftarrow 1$;
repeat
  foreach $i \in \mathcal{N}$;
  do
    $x_i^{(t+1)} = \max\left\{0,\ \frac{a_i - p}{b_i + c} + \sum_{j \neq i} \frac{g_{ij} - c}{b_i + c} x_j^{(t)}\right\}$;
  end
  $t \leftarrow t + 1$;
until $\|\mathbf{x}^{(t)} - \mathbf{x}^{(t-1)}\| \leq \epsilon$;
return $\mathbf{x}^{(t)}$;

Theorem 3. *Algorithm 1 computes the UDE in Stage II.*

The proof is given in Appendix and the main idea is to show that the best response updates in the algorithm result in a contraction mapping and hence converges to a fixed point.

It is desirable for users to reach the UDE in a distributed manner. We then propose a distributed update algorithm based on Algorithm 1, as described in Algorithm 2.

Note that the usage update in Algorithm 2 is equivalent to the best response update in Algorithm 1. Each user i chooses its best response usage based on the usage of its

Algorithm 2: Distributed algorithm to achieve the UDE in Stage II

each user $i \in \mathcal{N}$ **chooses** an initial usage $x_i^{(0)} \geq 0$;
loop at each time interval $t = 1, 2, \ldots$
each user $i \in \mathcal{N}$ in parallel:
updates its usage by

$$\max\left\{0, \; \frac{a_i - p}{b_i + c} + \frac{1}{b_i + c}\sum_{j \neq i, g_{ij} > 0} g_{ij} x_j^{(t)} - \frac{c}{b_i + c}\sum_{j \neq i} x_j^{(t)}\right\}$$

end loop

social friends who have social influences to it (i.e., each user j with $g_{ij} > 0$), which can be obtained from the social friends, and the total usage of all users, which can be obtained from the wireless provider. The correctness of Algorithm 2 follows from that of Algorithm 1 and is thus omitted.

PROPOSITION 1. *Algorithm 2 achieves the UDE in Stage II.*

3.3 Parameter Analysis at UDE

We first investigate the impact of price on the UDE. To draw clean insights, we start with the case for two users. Without loss of generality, assume that $a_1 \geq a_2$.

PROPOSITION 2. *For Stage II for two users, there exists a price threshold $p_{th} \in [0, a_1]$ where*

$$p_{th} = \frac{a_2(b_1 + c) - a_1(c - g_{21})}{b_1 + g_{21}} \tag{3}$$

such that the UDE $\mathbf{x}^*$ is given as follows, depending on the price p:

- High price regime: *When $p \geq a_1$, $x_1^* = x_2^* = 0$;*
- Medium price regime: *When $p_{th} \leq p < a_1$, $x_1^* = \frac{a_1 - p}{b_1 + c}$ and $x_2^* = 0$;*
- Low price regime: *When $0 \leq p < p_{th}$, $x_1^* = \frac{(a_1-p)(b_2+c)-(a_2-p)(c-g_{12})}{b_1 b_2 - g_{12} g_{21} + c(b_1+b_2+g_{12}+g_{21})}$ and $x_2^* = \frac{(a_2-p)(b_1+c)-(a_1-p)(c-g_{21})}{b_1 b_2 - g_{12} g_{21} + c(b_1+b_2+g_{12}+g_{21})}$.*

Due to space limitation, the proof is given in our online technical report [9]. According to (3), there are three cases of the threshold p_{th} depending on which effect dominates user 2's experience from user 1 (as illustrated in Fig. 3).

1. *Neither effect:* When $g_{21} = c$, we have $p_{th} = a_2$. As network effect and congestion effect cancel each other, user 2 experiences neither effect from user 1. Then user 2's usage demand is equal to its internal demand, and it reaches 0 when $p = a_2$.
2. *Congestion effect:* When $g_{21} < c$, we have $p_{th} < a_2$. As user 2 experiences congestion effect from user 1, even when p is less than a_2 such that user 2 has a positive internal demand, its external demand can be sufficiently negative such that user 2's usage demand is negative. In particular, when $\frac{a_1}{a_2} \geq \frac{b_1+c}{c-g_{21}}$, user 2's usage is 0 even when the price is 0.

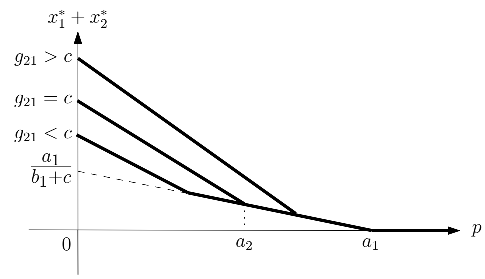


Figure 3: Total usage at the UDE in Stage II for two users.

3. *Network effect:* When $g_{21} > c$, we have $p_{th} > a_2$. As user 2 experiences network effect from user 1, even when p is greater than a_2 such that user 2 has a negative internal demand, its external demand can be sufficiently positive such that user 2's usage demand is positive.

Next we study the general case for any number of users. For convenience, let us define

$$B = \begin{bmatrix} b_1 + c & c & \cdots & c \\ c & b_2 + c & \cdots & c \\ \vdots & \vdots & \ddots & \vdots \\ c & c & \cdots & b_N + c \end{bmatrix}, G = \begin{bmatrix} 0 & g_{12} & \cdots & g_{1N} \\ g_{21} & 0 & \cdots & g_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N1} & g_{N2} & \cdots & 0 \end{bmatrix}.$$

Also define C as the $N \times N$ matrix with each entry being c. For a UDE $\mathbf{x}^*$, let $\mathcal{S}$ be the set of users with positive usage in $\mathbf{x}^*$ (i.e., $x_i^* > 0$, $\forall i \in \mathcal{S}$ and $x_i^* = 0$, $\forall i \notin \mathcal{S}$). For convenience, let $\mathbf{v}_\mathcal{S}$ denote the $|\mathcal{S}| \times 1$ vector comprised of the entries of a vector $\mathbf{v}$ with indices in $\mathcal{S}$, $M_\mathcal{S}$ denote the $|\mathcal{S}| \times |\mathcal{S}|$ matrix comprised of the entries of a matrix M with indices in $\mathcal{S} \times \mathcal{S}$, and $[M]_{i,\mathcal{S}}$ denote the $1 \times |\mathcal{S}|$ vector comprised of the entries of the ith row of a matrix M with column indices in $\mathcal{S}$. According to the best response function (2), $\mathbf{x}_\mathcal{S}^*$ is the solution to the system of equations

$$B_\mathcal{S} \mathbf{x}_\mathcal{S} = \mathbf{a}_\mathcal{S} - p\mathbf{1}_\mathcal{S} + G_\mathcal{S} \mathbf{x}_\mathcal{S}$$

where $\mathbf{1}$ denotes the $N \times 1$ vector of 1s. We need the following lemma:

LEMMA 1. *Under Assumption 1, $(B_\mathcal{S} - G_\mathcal{S})$ is invertible for any set $\mathcal{S} \subseteq \mathcal{N}$.*

The proof is given in [9]. Thus we have

$$\mathbf{x}_\mathcal{S}^* = (B_\mathcal{S} - G_\mathcal{S})^{-1}(\mathbf{a}_\mathcal{S} - p\mathbf{1}_\mathcal{S}). \tag{4}$$

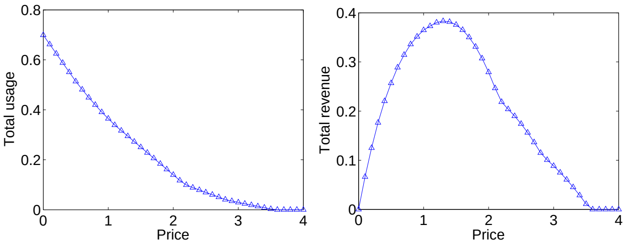
When users have the same internal coefficient a_i, we can show that the same set of users have positive equilibrium usage at different prices.

PROPOSITION 3. *For Stage II, when $a_i = a$, $\forall i$, the UDE $\mathbf{x}^*$ is given as follows, depending on the price p:*

- *When $p > a$, $x_i^* = 0$, $\forall i$;*
- *When $0 \leq p \leq a$, there exists a set $\mathcal{S} \subseteq \mathcal{N}$ such that for any $p \in [0, a)$,*

$$x_i^* = [(B_\mathcal{S} - G_\mathcal{S})^{-1}(\mathbf{a}_\mathcal{S} - p\mathbf{1}_\mathcal{S})]_i > 0, \forall i \in \mathcal{S},$$

and $x_i^* = 0$, $\forall i \notin \mathcal{S}$, where $[M]_i$ denotes the ith row of matrix M.



(a) Total usage vs. price p. (b) Total revenue vs. price p.

Figure 4: (a) Total usage at the UDE is a piece-wise linear function of price; (b) Total revenue at the UDE is a piece-wise quadratic function of price.

The proof is given in Appendix. Proposition 3 shows that the set of users with positive equilibrium usage (if they exist) does not change with price, and each user's positive usage decreases when price increases.

We then show by a counterexample that if users have different internal coefficients a_i, a user's equilibrium usage can increase when price increases. Consider a case for three users where $a_1 = 2$, $a_2 = a_3 = 1.5$, $b_i = 3$, $\forall i$, $c = 2$, $p = 0.4$, and $g_{23} = g_{32} = 4$, $g_{ij} = 0$, $\forall \{i, j\} \neq \{2, 3\}$[2]. We can show that there exists a unique UDE $\mathbf{x}^*$ and it is the solution to the system of equations below:

$$5x_1 + 2x_2 + 2x_3 = 1.6$$
$$5x_2 + 2x_1 - 2x_3 = 1.1$$
$$5x_3 - 2x_2 + 2x_1 = 1.1.$$

Solving these equations, we have $x_1^* = 0.0571$, $x_2^* = 0.3286$, $x_3^* = 0.3286$. When price p increases to 0.5, the new UDE is the solution to

$$5x_1 + 2x_2 + 2x_3 = 1.5$$
$$5x_2 + 2x_1 - 2x_3 = 1$$
$$5x_3 - 2x_2 + 2x_1 = 1$$

which is $x_1^* = 0.0714 > 0.0571$, $x_2^* = 0.2857$, $x_3^* = 0.2857$. Thus the usage of user 1 increases.

Remark: Intuitively, when the price increases, the usage of both user 2 and 3 decrease and the internal demand of user 1 decreases. However, as user 1 experiences strong congestion effect from both user 2 and 3, user 1's external demand increases due to the decrease of congestion effect, and it increases faster than the decrease of user 1's internal demand as price increases, such that the total of internal and external demand increases. In addition, a larger internal coefficient a_1 of user 1 than that of user 2 and 3 allows user 1 to have a positive equilibrium usage $x_1^* = 0.0714$ even when its external demand is negative due to the strong congestion effect. Indeed, if user 1 has the same internal coefficient $a_1 = 1.5$ as user 2 and 3, then we can show that its equilibrium usage is 0.

Furthermore, when users have different internal coefficients a_i, we have the following result.

PROPOSITION 4. *For Stage II, the UDE $\mathbf{x}^*$ is given as follows, depending on the price p:*

- *When $p > \max_{i \in \mathcal{N}} a_i$, $x_i^* = 0$, $\forall i$;*
- *When $0 \leq p \leq \max_{i \in \mathcal{N}} a_i$, there is a set of prices $p_0 \triangleq 0 < p_1 < \cdots < p_M < p_{M+1} \triangleq \max_{i \in \mathcal{N}} a_i$, and for each $k \in \{0, \ldots, M\}$, there exists a set $\mathcal{S}_k \subseteq \mathcal{N}$ such that for any $p \in [p_k, p_{k+1}]$,*
$$x_i^* = [(B_{\mathcal{S}_k} - G_{\mathcal{S}_k})^{-1}(\mathbf{a}_{\mathcal{S}_k} - p\mathbf{1}_{\mathcal{S}_k})]_i > 0, \forall i \in \mathcal{S}_k$$
and $x_i^* = 0$, $\forall i \notin \mathcal{S}_k$.

[2]Note that Assumption 1 holds under this setting.

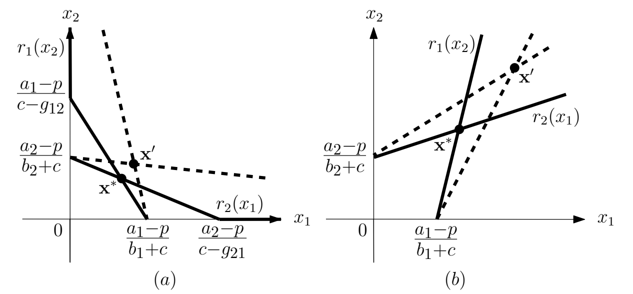


Figure 5: For Stage II for two users, the unique UDE $\mathbf{x}^*$ is achieved at the intersection of the best response functions (bold lines): (a) $\frac{g_{12}-c}{b_1+c} \in (-1, 0)$, $\frac{g_{21}-c}{b_2+c} \in (-1, 0)$; (b) $\frac{g_{12}-c}{b_1+c} \in (0, 1)$, $\frac{g_{21}-c}{b_2+c} \in (0, 1)$. When $g_{12} = g_{21}$ increases, the UDE $\mathbf{x}^*$ moves to $\mathbf{x}'$.

The proof is given in [9]. Proposition 4 shows that each user's equilibrium usage is a piece-wise linear function of price: within each price interval $[p_k, p_{k+1}]$, the equilibrium usage is a linear function of price p.

Next we investigate the impacts of other parameters on the UDE. We show that the total usage always increases when a user's parameter improves, under the condition that users' social network is symmetric. For tractable analysis, we assume that users have the same internal coefficient a_i[3].

PROPOSITION 5. *For Stage II, when $a_i = a$, $\forall i$ and social ties are symmetric (i.e., $g_{ij} = g_{ji}$, $\forall i \neq j$), the total equilibrium usage increases when a or any g_{ij} increases, or any b_i or c decreases.*

The proof is given in Appendix. We illustrate Proposition 5 by an example in Fig. 5. As mentioned before, users' social ties tend to be symmetric in practice due to social reciprocity [8]. In Section 5, simulation results will show that the performance under asymmetric social ties is very close to that under symmetric social ties.

4. STAGE I: OPTIMAL PRICING

In the previous section, we have investigated the UDE in Stage II given a price chosen by the wireless provider. In this section, we study the optimal pricing of the provider in Stage I.

We first observe from Proposition 4 that the total usage is a piece-wise linear function of price (as illustrated in Fig. 4(a)). As a result, the total revenue is a piece-wise quadratic function of price (as illustrated in Fig. 4(b)). Based on this observation, we develop an algorithm that computes the optimal price to maximize the provider's revenue, as described in Algorithm 3. The basic idea is to first

[3]Users can still have different b_i.

Algorithm 3: Compute the optimal price that maximizes revenue in Stage I

compute the UDE $\mathbf{x}^*$ at price 0 using Algorithm 1;
$\underline{p} \leftarrow 0$; $p^* \leftarrow 0$; $r^* \leftarrow 0$; $\mathcal{S} \leftarrow \varnothing$;
foreach $i \in \mathcal{N}$ **do**
  if $x_i^* > 0$ **then**
    $\mathcal{S} \leftarrow \mathcal{S} \cup \{i\}$;
  end
end
while $\underline{p} \leq \max_{i\in\mathcal{N}} a_i$ *and* $\mathcal{S} \neq \varnothing$ **do**
  $\mathcal{S}' \leftarrow \varnothing$; $\mathcal{S}'' \leftarrow \varnothing$;
  foreach $i \in \mathcal{S}$ **do**
    if $[(B_\mathcal{S} - G_\mathcal{S})^{-1}]_i \mathbf{1}_\mathcal{S} > 0$ **then**
      $\mathcal{S}' \leftarrow \mathcal{S}' \cup \{i\}$; $\tilde{p}_i \leftarrow \frac{[(B_\mathcal{S}-G_\mathcal{S})^{-1}]_i \mathbf{a}_\mathcal{S}}{[(B_\mathcal{S}-G_\mathcal{S})^{-1}]_i \mathbf{1}_\mathcal{S}}$;
    end
  end
  foreach $i \notin \mathcal{S}$ **do**
    if $[G - C]_{i,\mathcal{S}}(B_\mathcal{S} - G_\mathcal{S})^{-1}\mathbf{1}_\mathcal{S} < -1$ **then**
      $\mathcal{S}'' \leftarrow \mathcal{S}'' \cup \{i\}$;
      $\tilde{p}_i \leftarrow \frac{[G-C]_{i,\mathcal{S}}(B_\mathcal{S}-G_\mathcal{S})^{-1}\mathbf{a}_\mathcal{S}+a_i}{[G-C]_{i,\mathcal{S}}(B_\mathcal{S}-G_\mathcal{S})^{-1}\mathbf{1}_\mathcal{S}+1}$;
    end
  end
  $\overline{p} \leftarrow \min_{i\in\mathcal{S}'\cup\mathcal{S}''} \tilde{p}_i$; $k \leftarrow \arg\min_{i\in\mathcal{S}'\cup\mathcal{S}''} \tilde{p}_i$;
  $\hat{p} \leftarrow \frac{\mathbf{1}_\mathcal{S}^T(B_\mathcal{S}-G_\mathcal{S})^{-1}\mathbf{a}_\mathcal{S}}{2\mathbf{1}_\mathcal{S}^T(B_\mathcal{S}-G_\mathcal{S})^{-1}\mathbf{1}_\mathcal{S}}$;
  if $\hat{p} \in [\underline{p}, \overline{p}]$ **then**
    $p' \leftarrow \hat{p}$;
    else
      if $\hat{p} < \underline{p}$ **then**
        $p' \leftarrow \underline{p}$;
        else
          $p' \leftarrow \overline{p}$;
        end
      end
    end
  end
  $r' \leftarrow p'\mathbf{1}_\mathcal{S}^T(B_\mathcal{S} - G_\mathcal{S})^{-1}(\mathbf{a}_\mathcal{S} - p'\mathbf{1}_\mathcal{S})$;
  if $r' > r^*$ **then**
    $p^* \leftarrow p'$; $r^* \leftarrow r'$;
  end
  $\underline{p} \leftarrow \overline{p}$;
  if $k \in \mathcal{S}$ **then**
    $\mathcal{S} \leftarrow \mathcal{S} \setminus \{k\}$;
    else
      $\mathcal{S} \leftarrow \mathcal{S} \cup \{k\}$;
    end
  end
end
return p^*, r^*;

determine the price intervals that characterize the piece-wise structure, such that within each price interval, the set of users with positive usage is the same at any price. Then we find the optimal price within each interval that maximizes the revenue. Thus we can find the optimal price with the maximum revenue among all the intervals.

In particular, Algorithm 3 starts with computing the set of users $\mathcal{S}$ with positive usage at price 0 by using Algorithm 1. Then this set $\mathcal{S}$ serves as the initial condition for the following steps. As the price p increases from 0 to $\max_{i\in\mathcal{N}} a_i$ (which is the largest possible value of the optimal price according to Theorem 4), it iteratively finds the critical prices at which the set $\mathcal{S}$ changes. In each iteration, given the current critical price $\underline{p}$, the next critical price $\overline{p}$ is the minimum price greater than $\underline{p}$ at which some user $i \in \mathcal{S}$ with positive usage decreases its usage to 0, or some user $i \notin \mathcal{S}$ with usage 0 increases its usage to a positive value[4]. Within each price interval $[\underline{p}, \overline{p}]$, as the revenue R is a quadratic function of price p, the optimal price in $[\underline{p}, \overline{p}]$ that maximizes the revenue is the price $\hat{p}$ such that $\frac{\partial R(p)}{\partial p}|_{p=\hat{p}} = 0$, if $\hat{p}$ is in $[\underline{p}, \overline{p}]$; otherwise, the optimal price is one of the endpoints $\underline{p}$ and $\overline{p}$. By comparing the maximum revenues at the optimal prices for all the price intervals, the algorithm finds the optimal price in the entire range of price.

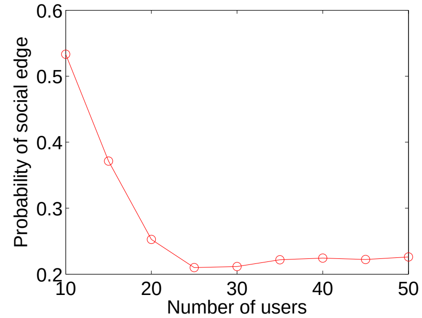


Figure 6: Probability of social edge vs. number of users in real data trace [10].

THEOREM 4. *Algorithm 3 computes the optimal price in Stage I.*

The proof is given in [9]. In the next section, numerical results will show that the computational complexity of Algorithm 3 is linear in the number of users.

5. PERFORMANCE EVALUATION

In this section, we first use simulation results to evaluate the performance of the two-stage game for the mobile users and the wireless provider. Then we discuss the engineering insights that can be drawn from the simulation results.

5.1 Simulation Setup

To illustrate the impacts of different parameters of mobile social networks on the performance, we consider a random setting as follows. We simulate the social graph G using the Erdős-Rényi (ER) graph model [11], where a social edge exists between each pair of users with probability P_S. If a social edge exists, the social tie follows a normal distribution $N(\mu_G, 2)$ (with mean μ_G and variance 2). We assume that each a_i follows a normal distribution $N(\mu_A, 2)$, and each b_i follows a normal distribution $N(\mu_B, 2)$. We set default parameter values as follows: $N = 10$, $P_s = 0.8$, $\mu_A = 4$, $\mu_B = 10$, $\mu_G = 4$, $c = 4$. To evaluate the performance in practice, we also simulate the social graph according to

[4]Recall that some user's equilibrium usage can increase when price increases as illustrated by the example in Section 3.

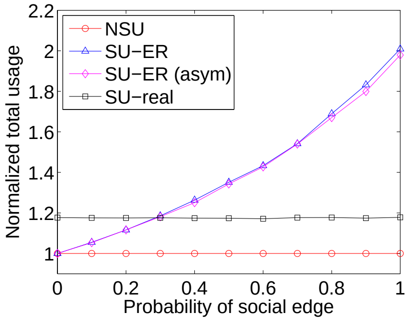


Figure 7: Normalized total usage vs. probability of social edge P_S.

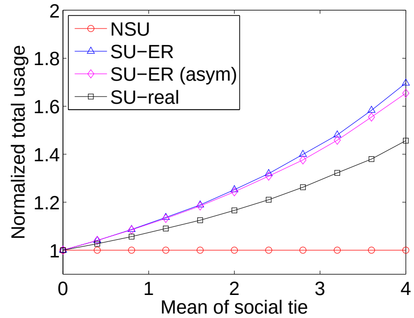


Figure 8: Normalized total usage vs. mean of social tie μ_G.

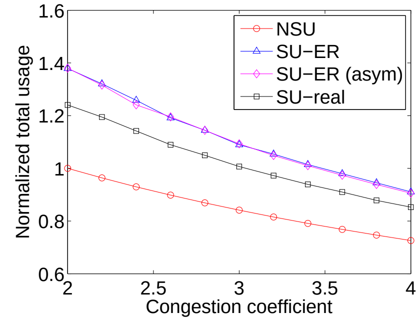


Figure 9: Normalized total usage vs. congestion coefficient c.

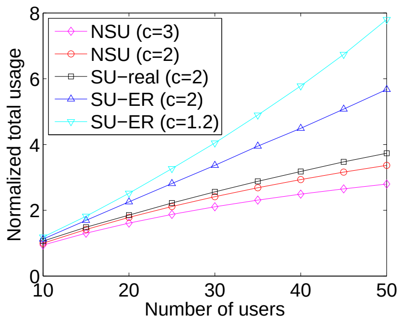


Figure 10: Normalized total usage vs. number of users N.

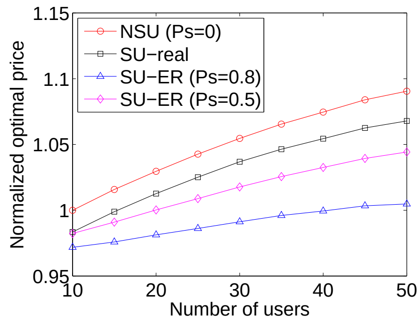


Figure 11: Normalized optimal price vs. number of users N.

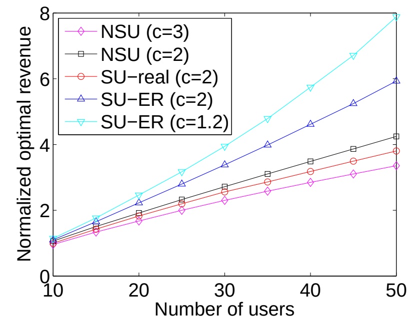


Figure 12: Normalized optimal revenue vs. number of users N.

the real data trace from Brightkite [10], which is a social friendship network based on mobile phones. For this data trace, we plot the average number of social ties between two users versus the number of users in Fig. 6. If a social edge exists based on the real data, the social tie also follows a normal distribution $N(\mu_G, 2)$.

As a benchmark, we evaluate the performance when users demand non-socially-aware usage (NSU) in comparison to our proposed socially-aware usage (SU). Since NSU is a special case of SU with all social ties being 0, the UDE and optimal pricing for NSU can be computed as for SU. To highlight the performance comparison, we normalize the results with respect to NSU. We also compare the performance under SU with ER model based social graph (SU-ER) and with real data based social graph (SU-real).

5.2 Simulation Results

5.2.1 *Total Usage in Stage II*

We first evaluate the performance of total usage in Stage II.

We illustrate the impacts of P_S, μ_G, c on total usage in Figs. 7-9, respectively. As expected, we observe from all these figures that SU always dominates NSU, and can perform significantly better than NSU. From Figs. 7-8, we can see that the performance gain of SU over NSU increases as P_S or μ_G increases, and the marginal gain is also increasing. Similarly, we can see from Fig. 9 that the performance gain of SU over NSU increases as congestion coefficient c decreases, and the marginal gain is also increasing. We also evaluate the performance under SU with ER model based the asymmetric social graph. We observe that its performance is very close to that with the symmetric social graph.

Fig. 10 illustrates the impact of N on total usage. As expected, we observe that the total usage always increases with the number of users. However, for the case of NSU and SU-real, the marginal gain of total usage decreases with the number of users, while for the case of SU-ER, the marginal gain increases. Intuitively, in the former case, when a new user joins the network, as the new user's social ties with the existing users are weak, the congestion effect between the new user and the existing users outweighs the network effect between them. Furthermore, as more users exist in the network, the weight difference between the congestion effect and the network effect increases, and thus the marginal gain of total usage by adding a user decreases. In the latter case, as the new user's social ties with the existing users are strong, the roles of the congestion effect and network effect are switched.

5.2.2 *Optimal Price in Stage I*

Next we evaluate the performance of the optimal price and optimal revenue in Stage I.

Fig. 11 illustrates the optimal price as the number of users increases. We observe that the optimal price always decreases with the number of users. Intuitively, this is because as the number of users increases, more users have higher internal demands, so that increasing the price does not result in significant decrease in total usage. Comparing different curves, we can also see that the optimal price decreases as P_S increases from 0 to 0.3 and then to 0.8. Intuitively, this is because that when network effect is strong, a low price

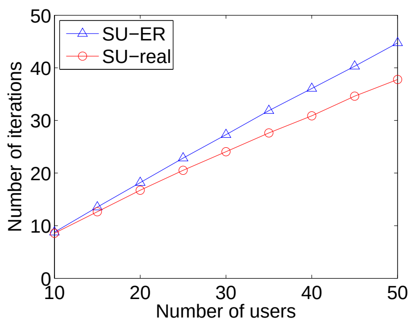


Figure 13: Computational complexity of Algorithm 3 vs. number of users N.

is desirable, since it encourages users' internal usage which further stimulate significantly more usage by the network effect; when congestion effect is strong, a high price is desirable, since decreasing the price cannot encourage significantly more usage due to the congestion effect.

Fig. 12 illustrates the optimal revenue achieved at the optimal price as the number of users increases. As expected, we can make similar observations as for Fig. 10: when network effect dominates congestion effect, the marginal gain of optimal revenue by adding more users is increasing; otherwise, the marginal gain is decreasing.

Fig. 13 illustrates the computational complexity of Algorithm 3 as the number of users increases. The number of iterations is equal to the number of price intervals that determine the piece-wise structure of total usage and revenue as a function of price. We observe that the complexity is $O(N)$.

5.3 Further Discussions

Based on the simulation results, we can draw the following engineering insights for the operation of wireless providers.

- The observations from Figs. 7-9 suggest that as users' social ties become stronger (which can be promoted by social services), the wireless provider can receive an increasing total usage and thus revenue, and also an increasing marginal gain. In addition, the wireless provider can also receive an increasing marginal return by incorporating more resources for the wireless service to mitigate congestion.
- The observations from Figs. 10 and 12 suggest that the wireless provider should be aware of whether the network effect determined by users' social ties dominates the congestion effect. If the network effect dominates, it receives an increasing marginal gain by taking in more users; otherwise, the marginal gain is decreasing and the total usage will saturate when the number of users is sufficiently large.
- The observations from Figs. 11 suggest that the wireless provider should set a low price when users' social ties are strong (evidenced by the popularity of social services), as the decrease of price will be outweighed by the increase of total usage resulted from the network effect, so that the total revenue increases. Otherwise, the wireless provider should set a high price, as cutting the price cannot stimulate sufficiently more usage due to the congestion effect to compensate the price decrease.

6. RELATED WORK

There have been many studies on users' behaviors and the provider's pricing strategy when either network effect (also known as positive externality) or congestion effect is present, respectively [5,12,13]. In [5], different pricing strategies of a provider have been studied where users' behaviors are only subject to network effect. When users experience both network effect and congestion effect as considered in this paper, the coupling among users is very different and more complex than when only network effect is present as in [5]. Very few work have studied the case where both network effect and congestion effect are present. [14] has studied users' behaviors when they experience both network effect and congestion effect. However, it assumes that the network effect is the same for all users, which does not capture the fact that users experience different levels of network effect based on their diverse social ties as considered in this paper.

The social aspect of mobile networking is an emerging paradigm for network design and optimization. Social contact patterns have been exploited for efficient data forwarding and dissemination in delay tolerant networks [15,16]. Social trust and social reciprocity have been leveraged in [17] to enhance cooperative D2D communication based on a coalitional game. A social group utility maximization (SGUM) framework has been recently studied in [18–20], which captures the impact of mobile users' diverse social ties on the interactions of their mobile devices subject to diverse physical relationships.

7. CONCLUSION

In this paper, we have formulated the interaction between mobile users and a wireless provider as a Stackelberg game, by jointly considering the network effect in the social domain and the congestion effect in the physical wireless domain. For Stage II, we have analyzed users' demand equilibrium given a price chosen by the wireless provider. For Stage I, we have developed an algorithm to compute the optimal price to maximize the wireless provider's revenue. We have also conducted simulations using real data to evaluate the performance, and drawn useful engineering insights for the operation of wireless providers.

For future work, we can examine other utility functions, e.g., a logarithmic function for internal utility, yet the major engineering insights should remain the same. Another interesting direction is to study the provider's pricing strategy when it is allowed to differentiate the price for different users. In this case, the price offered to each user will depend on its social influences to others based on the social network.

ACKNOWLEDGEMENT

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APPENDIX

Proof of Theorem 1

To show the existence of UDE, we make use of the following lemma, which shows that the Stage II game with unbounded usage range is equivalent to that with bounded usage range.

LEMMA 2. *Under Assumption 1, the Stage II game* $\mathcal{G} \triangleq \{\mathcal{N}, \{u_i\}_{i\in\mathcal{N}}, [0,\infty)^N\}$ *admits the same set of UDEs as the game* $\mathcal{G}' \triangleq \{\mathcal{N}, \{u_i\}_{i\in\mathcal{N}}, [0,\bar{x}]^N\}$*, where* $\bar{x}$ *is any number that satisfies* $\bar{x} > \max_{i\in\mathcal{N}} |a_i - p|/(b_i + c - \sum_{j\neq i} |g_{ij} - c|)$.

Proof: Let $\mathbf{x}^*$ be any UDE of game $\mathcal{G}$ and x_i^* be the largest in $\mathbf{x}^*$, i.e., $x_i^* \geq x_j^*$, $\forall i \neq j$. If $x_i^* > 0$, using the best response function (2), we have

$$x_i^* = \frac{a_i - p}{b_i + c} + \sum_{j\neq i} \frac{g_{ij} - c}{b_i + c} x_j^* \leq \frac{|a_i - p|}{b_i + c} + \sum_{j\neq i} \frac{|g_{ij} - c|}{b_i + c} x_i^*. \tag{5}$$

Using Assumption 1, it follows from (5) that

$$x_i^* \leq |a_i - p|/(b_i + c - \sum_{j\neq i} |g_{ij} - c|) < \bar{x}.$$

Since x_i^* is the largest in $\mathbf{x}^*$, we have $x_j^* \in [0, \bar{x}]$, $\forall j \in \mathcal{N}$, and thus $\mathbf{x}^* \in [0, \bar{x}]^N$. Therefore, as game $\mathcal{G}$ and game $\mathcal{G}'$ have the same set of payoff functions and the strategy spaces in both games contain $[0, \bar{x}]^N$, they have the same set of UDEs. □

Using a celebrated result in [21–23], the infinite game $\mathcal{G}'$ admits a UDE if the strategy space $[0, \bar{x}]^N$ is compact and convex, the payoff function $u_i(x_i, \mathbf{x}_{-i})$ is continuous in x_i and $\mathbf{x}_{-i}$, and the payoff function $u_i(x_i, \mathbf{x}_{-i})$ is concave in x_i. It is easy to check that all these conditions hold, and thus the game $\mathcal{G}'$ admits a UDE. Then it follows from Lemma 2 that the Stage II game $\mathcal{G}$ admits a UDE.

Proof of Theorem 2

We will show that the UDE is unique by showing that the game $\mathcal{G}'$ defined in Lemma 2 is a concave game. The Jacobian matrix $\nabla u(\mathbf{x})$ of the payoff function profile $u(\mathbf{x}) \triangleq$

$(u_1(\mathbf{x}),\ldots,u_N(\mathbf{x}))$ of game $\mathcal{G}'$ is given by

$$\nabla u(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 u_1(\mathbf{x})}{\partial x_1^2} & \frac{\partial^2 u_1(\mathbf{x})}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 u_1(\mathbf{x})}{\partial x_1 \partial x_N} \\ \frac{\partial^2 u_2(\mathbf{x})}{\partial x_2 \partial x_1} & \frac{\partial^2 u_2(\mathbf{x})}{\partial x_2^2} & \cdots & \frac{\partial^2 u_2(\mathbf{x})}{\partial x_2 \partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 u_N(\mathbf{x})}{\partial x_N \partial x_1} & \frac{\partial^2 u_N(\mathbf{x})}{\partial x_N \partial x_2} & \cdots & \frac{\partial^2 u_N(\mathbf{x})}{\partial x_N^2} \end{bmatrix}$$

$$= \begin{bmatrix} -b_1 - c & g_{12} - c & \cdots & g_{1N} - c \\ g_{21} - c & -b_2 - c & \cdots & g_{2N} - c \\ \vdots & \vdots & \ddots & \vdots \\ g_{N1} - c & g_{N2} - c & \cdots & -b_N - c \end{bmatrix}$$

$$= -(B - G).$$

Using Assumption 1, it follows that

$$[B - G]_{ii} \geq \sum_{j \neq i} |[B - G]_{ij}|, \ \forall i$$

where $[M]_{ij}$ denotes the entry in the ith row and jth column of matrix M. Therefore, $B-G$ is strictly diagonal dominant [24]. It follows from the condition $\sum_{j\neq i} \frac{|g_{ji}-c|}{b_i+c} < 1, \forall i$ that $(B-G)^T$ is also strictly diagonal dominant. Then we have that

$$\nabla u(\mathbf{x}) + \nabla u(\mathbf{x})^T = -(B - G) - (B - G)^T$$

is strictly diagonal dominant. Also observe that it is symmetric. It is known that a symmetric matrix that is strictly diagonally dominant with real nonnegative diagonal entries is positive definite [24]. Therefore, $\nabla u(\mathbf{x}) + \nabla u(\mathbf{x})^T$ is negative definite. It follows from [7, Theorem 6] that $u(\mathbf{x})$ is diagonally strictly concave. Therefore, using [7, Theorem 2], game $\mathcal{G}'$ has a unique UDE.

Proof of Theorem 3

Let $\Delta x_i^{(t)} \triangleq x_i^{(t)} - x_i^*, \forall i$. For any $i \in \mathcal{N}$, according to step 6 in Algorithm 1, we have

$$|\Delta x_i^{(t+1)}| \leq |\sum_{j\neq i} \frac{g_{ij} - c}{b_i + c} \Delta x_j^{(t)}| \leq \sum_{j \neq i} \frac{|g_{ij} - c|}{b_i + c} |\Delta x_j^{(t)}|. \quad (6)$$

Let $\|\Delta \mathbf{x}^{(t)}\|_\infty$ be the l_∞-norm of vector $(\Delta x_i^{(t)}, \ldots, \Delta x_N^{(t)})$, i.e.,

$$\|\Delta \mathbf{x}^{(t)}\|_\infty \triangleq \max_{i \in \mathcal{N}} |\Delta x_i^{(t)}|.$$

Then, using Assumption 1 and (6), we have

$$\begin{aligned} \|\Delta \mathbf{x}^{(t+1)}\|_\infty &\leq \max_{i\in\mathcal{N}} \left(\sum_{j\neq i} \frac{|g_{ij} - c|}{b_i + c} |\Delta x_j^{(t)}| \right) \\ &\leq \left(\max_{i\in\mathcal{N}} \sum_{j \neq i} \frac{|g_{ij} - c|}{b_i + c} \right) \left(\max_{i \in \mathcal{N}} |\Delta x_j^{(t)}| \right) \\ &= \left(\max_{i\in\mathcal{N}} \sum_{j \neq i} \frac{|g_{ij} - c|}{b_i + c} \right) \|\Delta \mathbf{x}^{(t)}\|_\infty. \end{aligned}$$

According to Assumption 1, we have $\left(\max_{i\in\mathcal{N}} \sum_{j\neq i} \frac{|g_{ij}-c|}{b_i+c}\right) < 1$. Then it follows that the algorithm results in a contraction mapping of $|\Delta x_i^{(t)}|$, and thus converges to the UDE.

Proof of Proposition 2

If the UDE is positive, i.e., $x_1^* > 0$ and $x_2^* > 0$, according to 2, we have $\mathbf{x}^* > 0$ is the solution to

$$x_1 = \frac{a_1 - p}{b_1 + c} + \frac{g_{12} - c}{b_1 + c} x_2, \ x_2 = \frac{a_2 - p}{b_2 + c} + \frac{g_{21} - c}{b_2 + c} x_1.$$

Solving it, we have the expression given in the low price regime. Then observe that x_1^* and x_2^* are both positive when $p = 0$, and decrease when p increases. Also observe that $x_1^* = 0$ when $p = p_1 \triangleq \frac{a_1(b_2+c)-a_2(c-g_{12})}{b_2+g_{12}}$, and $x_2^* = 0$ when $p = p_2 \triangleq \frac{a_2(b_1+c)-a_1(c-g_{21})}{b_1+g_{21}}$. We can check that $p_1 \geq p_2$. Therefore, when $p > p_2 = p_{th}$, we have $x_1^* > 0$ and $x_2^* = 0$. Thus $x_1^* = \frac{a_1-p}{b_1+c}$ according to (2). Then we further observe that $x_1^* = x_2^* = 0$ when $p > a_1$.

Proof of Lemma 1

We only prove the case when $\mathcal{S} = \mathcal{N}$, since the case when $\mathcal{S} \subset \mathcal{N}$ can be proved similarly. Let

$$\bar{B} = \begin{bmatrix} b_1 + c & 0 & \cdots & 0 \\ 0 & b_2 + c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_N + c \end{bmatrix},$$

$$\bar{G} = \begin{bmatrix} 0 & g_{12} - c & \cdots & g_{1N} - c \\ g_{21} - c & 0 & \cdots & g_{2N} - c \\ \vdots & \vdots & \ddots & \vdots \\ g_{N1} - c & g_{N2} - c & \cdots & 0 \end{bmatrix}.$$

Since $\bar{B}$ is a diagonal matrix with positive diagonal entries, it is invertible. Let λ be any eigenvalue of $\bar{B}^{-1}\bar{G}$ with v being the corresponding eigenvector. Let v_i be the largest entry of v in absolute value, i.e., $|v_i| \geq |v_j|, \forall j$. Since $(\bar{B}^{-1}\bar{G})v = \lambda v$, it follows that

$$\begin{aligned} |\lambda v_i| = |[\bar{B}^{-1}\bar{G}]_i v| &\leq \sum_{j \in \mathcal{N}} |[\bar{B}^{-1}\bar{G}]_{ij}||v_j| \\ &\leq |v_i| \sum_{j\in\mathcal{N}} \frac{|g_{ij} - c|}{b_i + c} < |v_i| \end{aligned}$$

where the last inequality follows from Assumption 1. It follows that the spectral radius of $\bar{B}^{-1}\bar{G}$ is strictly less than 1. Since each eigenvalue of $I - \bar{B}^{-1}\bar{G}$ is equal to $1-\lambda$ where λ is an eigenvalue of $\bar{B}^{-1}\bar{G}$, where I denotes the $N \times N$ identity matrix, it follows that $I - \bar{B}^{-1}\bar{G}$ has no eigenvalue of 0, and thus is invertible. Thus $B - G = \bar{B} - \bar{G} = \bar{B}(I - \bar{B}^{-1}\bar{G})$ is also invertible.

Proof of Proposition 3

We first show part 1). Suppose $p > a$ and $x_i^* > 0$ is the largest in $\mathbf{x}^*$, i.e., $x_i^* \geq x_j^*, \forall i \neq j$. Using the best response function (2), we have

$$x_i^* = \frac{a - p}{b_i + c} + \sum_{j \neq i} \frac{g_{ij} - c}{b_i + c} x_j^* \leq \sum_{j \neq i} \frac{|g_{ij} - c|}{b_i + c} x_i^* < x_i^*$$

where the last inequality follows from Assumption 1. This shows a contradiction. Thus we have $x_i^* = 0, \forall i$.

Next we show part 2). Let $\mathcal{S}$ be the set of users with positive usage in $\mathbf{x}^*$ at price 0. For any $i \notin \mathcal{S}$, using (2), we have

$$x_i^* = 0 \geq \frac{a}{b_i + c} + \frac{a}{b_i + c}[G - C]_{i,\mathcal{S}}(B_\mathcal{S} - G_\mathcal{S})^{-1}\mathbf{1}_\mathcal{S}. \quad (7)$$

For any $p \in (0, a]$, we next show that $\mathbf{x}'$ with $\mathbf{x}'_\mathcal{S} = (a - p)(B_\mathcal{S} - G_\mathcal{S})^{-1}\mathbf{1}_\mathcal{S}$ and $x'_i = 0$, $\forall i \notin \mathcal{S}$ is the UDE at price p. We observe that for any $i \in \mathcal{S}$, x'_i is its best response at $\mathbf{x}'$. For any $i \notin \mathcal{S}$, using (7), we have

$$\frac{a - p}{b_i + c} + \frac{a - p}{b_i + c}[G - C]_{i,\mathcal{S}}(B_\mathcal{S} - G_\mathcal{S})^{-1}\mathbf{1}_\mathcal{S} \leq 0 = x_i^*,$$

and thus is user i's best response at $\mathbf{x}'$.

Proof of Proposition 4

The proof of part 1) is the same as the proof of part 1) of Proposition 3 except that a should change to $\max_{i \in \mathcal{N}} a_i$. Now we show part 2). For any price $p \in [0, \max_{i \in \mathcal{N}} a_i]$, the usage of the set of users $\mathcal{S}$ with positive usage at the UDE (if they exist) is given by (4). Observe that the usage demand $\frac{a_i - p}{b_i + c} + \sum_{j \neq i} \frac{g_{ij} - c}{b_i + c} x_j^*$ of any user i at the UDE is a continuous function of price p and other users' usage x_j^*. Therefore, when the price p increases by a sufficiently small amount to p', the set of user with positive usage at the UDE is still the set $\mathcal{S}$, and thus their usage is still given by (4) except with p replaced by p'. Therefore, the set of user with positive usage is the same at any price in a continuous price interval. Then the desired result follows.

Proof of Proposition 5

We only prove the case when any g_{ij} increases, since the cases when a increases, any b_i decreases, or c increases can be proved similarly. Then it suffices to prove the case when any g_{ij} increases by any small amount. Let G be a symmetric matrix. Let $\mathbf{x}^*$ be the UDE under G and $\mathcal{S}$ be the set of users with positive usage in $\mathbf{x}^*$. It is easy to check that the UDE is a continuous function of the matrix G. Then we can always find a symmetric matrix G' with $[G']_{ij} \geq [G]_{ij}$, $\forall i, j$ and at least one strict inequality, such that the set of users with positive usage at the UDE $\mathbf{x}'$ under G' is also $\mathcal{S}$. Therefore, using the best response functions (2), we have

$$B_\mathcal{S}\mathbf{x}^*_\mathcal{S} = (a - p)\mathbf{1}_\mathcal{S} + G_\mathcal{S}\mathbf{x}^*_\mathcal{S} \quad (8)$$
$$B_\mathcal{S}\mathbf{x}'_\mathcal{S} = (a - p)\mathbf{1}_\mathcal{S} + G'_\mathcal{S}\mathbf{x}'_\mathcal{S}. \quad (9)$$

Subtracting (8) from (9), we have

$$B_\mathcal{S}(\mathbf{x}'_\mathcal{S} - \mathbf{x}^*_\mathcal{S}) = G_\mathcal{S}(\mathbf{x}'_\mathcal{S} - \mathbf{x}^*_\mathcal{S}) + \Delta G_\mathcal{S}\mathbf{x}'_\mathcal{S} \quad (10)$$

where $\Delta G_\mathcal{S} \triangleq G'_\mathcal{S} - G_\mathcal{S}$. According to Lemma 1, $B_\mathcal{S} - G_\mathcal{S}$ is invertible. Then it follows from (10) that

$$\mathbf{x}'_\mathcal{S} - \mathbf{x}^*_\mathcal{S} = (B_\mathcal{S} - G_\mathcal{S})^{-1}\Delta G_\mathcal{S}\mathbf{x}'_\mathcal{S}. \quad (11)$$

On the other hand, it follows from (8) that

$$\mathbf{x}^*_\mathcal{S} = (a - p)(B_\mathcal{S} - G_\mathcal{S})^{-1}\mathbf{1}_\mathcal{S}. \quad (12)$$

Using (11) and (12), we have

$$\begin{aligned} t(\mathbf{x}') - t(\mathbf{x}^*) &= \mathbf{1}_\mathcal{S}^T(\mathbf{x}'_\mathcal{S} - \mathbf{x}^*_\mathcal{S}) \\ &= \mathbf{1}_\mathcal{S}^T(B_\mathcal{S} - G_\mathcal{S})^{-1}\Delta G_\mathcal{S}\mathbf{x}'_\mathcal{S} \\ &= [(B_\mathcal{S} - G_\mathcal{S})^{-1}\mathbf{1}_\mathcal{S}]^T\Delta G_\mathcal{S}\mathbf{x}'_\mathcal{S} \\ &= \frac{1}{a - p}(\mathbf{x}^*_\mathcal{S})^T\Delta G_\mathcal{S}\mathbf{x}'_\mathcal{S} \end{aligned}$$

where the third equality is due to the fact that $(B_\mathcal{S} - G_\mathcal{S})^{-1}$ is symmetric since $B_\mathcal{S} - G_\mathcal{S}$ is symmetric. Since $a > p$ and $\mathbf{x}^*_\mathcal{S}$, ΔG, $\mathbf{x}'_\mathcal{S}$ only have nonnegative entries, it follows that $t(\mathbf{x}') \geq t(\mathbf{x}^*)$.

Proof of Theorem 4

We will show that given the current critical price $\underline{p}$ and the set of users $\mathcal{S}$ with positive usage at the UDE at the price $\underline{p}$, each iteration from step 8 to step 43 finds the next critical price $\overline{p}$, and the optimal price and revenue in the price interval $[\underline{p}, \overline{p}]$. For any $i \in \mathcal{S}$ with $x_i^* > 0$, it follows from (4) that

$$x_i^* = [(B_\mathcal{S} - G_\mathcal{S})^{-1}]_i\mathbf{a}_\mathcal{S} - p[(B_\mathcal{S} - G_\mathcal{S})^{-1}]_i\mathbf{1}_\mathcal{S} > 0. \quad (13)$$

Therefore, x_i^* decreases when the price p increases if

$$[(B_\mathcal{S} - G_\mathcal{S})^{-1}]_i\mathbf{1}_\mathcal{S} > 0. \quad (14)$$

For any $i \notin \mathcal{S}$ with $x_i^* = 0$, it follows from (2) that the usage demand of user i is no greater than 0 such that

$$\begin{aligned} 0 &\geq [G - C]_{i,\mathcal{S}}(B_\mathcal{S} - G_\mathcal{S})^{-1}(\mathbf{a}_\mathcal{S} - p\mathbf{1}_\mathcal{S}) + a_i - p \\ &= [G - C]_{i,\mathcal{S}}(B_\mathcal{S} - G_\mathcal{S})^{-1}\mathbf{a}_\mathcal{S} + a_i \\ &\qquad - p([G - C]_{i,\mathcal{S}}(B_\mathcal{S} - G_\mathcal{S})^{-1}\mathbf{1}_\mathcal{S} + 1) \end{aligned} \quad (15)$$

Therefore, the usage demand of user $i \notin \mathcal{S}$ increases when the price p increases if

$$[G - C]_{i,\mathcal{S}}(B_\mathcal{S} - G_\mathcal{S})^{-1}\mathbf{1}_\mathcal{S} < -1. \quad (16)$$

Using (13), the price at which a user $i \in \mathcal{S}$ changes its usage from positive to 0 is

$$\tilde{p}_i \triangleq \frac{[(B_\mathcal{S} - G_\mathcal{S})^{-1}]_i\mathbf{a}_\mathcal{S}}{[(B_\mathcal{S} - G_\mathcal{S})^{-1}]_i\mathbf{1}_\mathcal{S}}$$

where $\mathcal{S}'$ is the set of users such that (14) holds. Using (15), the price at which a user $i \notin \mathcal{S}$ changes its usage from 0 to positive is

$$\tilde{p}_i \triangleq \frac{[G - C]_{i,\mathcal{S}}(B_\mathcal{S} - G_\mathcal{S})^{-1}\mathbf{a}_\mathcal{S} + a_i}{[G - C]_{i,\mathcal{S}}(B_\mathcal{S} - G_\mathcal{S})^{-1}\mathbf{1}_\mathcal{S} + 1}$$

where $\mathcal{S}''$ is the set of users such that (16) holds. Therefore, the next critical price $\overline{p}$ is

$$\overline{p} = \min_{i \in \mathcal{S}' \cup \mathcal{S}''} \tilde{p}_i$$

Using (4), the revenue R is given by

$$R(p) = p\mathbf{1}_\mathcal{S}^T\mathbf{x}^*_\mathcal{S} = p\mathbf{1}_\mathcal{S}^T(B_\mathcal{S} - G_\mathcal{S})^{-1}(\mathbf{a}_\mathcal{S} - p\mathbf{1}_\mathcal{S})$$

which is a concave quadratic function of p. By setting $\frac{\partial R(p)}{\partial p} = 0$, we obtain that the optimal price p' in the price interval $[\underline{p}, \overline{p}]$ that maximizes the revenue is

$$p' = \hat{p} \triangleq \frac{\mathbf{1}_\mathcal{S}^T(B_\mathcal{S} - G_\mathcal{S})^{-1}\mathbf{a}_\mathcal{S}}{2\mathbf{1}_\mathcal{S}^T(B_\mathcal{S} - G_\mathcal{S})^{-1}\mathbf{1}_\mathcal{S}} \quad (17)$$

if $\hat{p} \in [\underline{p}, \overline{p}]$. If $\hat{p} \notin [\underline{p}, \overline{p}]$, the optimal price is $p' = \underline{p}$ if $\hat{p} < \underline{p}$, or $p' = \overline{p}$ if $\hat{p} > \overline{p}$. Thus the optimal revenue r' in the price interval $[\underline{p}, \overline{p}]$ is

$$r' = p'\mathbf{1}_\mathcal{S}^T(B_\mathcal{S} - G_\mathcal{S})^{-1}(\mathbf{a}_\mathcal{S} - p'\mathbf{1}_\mathcal{S}).$$

Then the optimal price and revenue in the entire range $[0, \max_{i \in \mathcal{N}} a_i]$ is found by comparing the optimal revenue for all the iterations from step 8 to step 43.